THE INTERNATIONAL CONGRESS OF MATHEMATICIANS

The next International Congress of Mathematicians will be held in Cambridge, Massachusetts, from August 30 to September 6, 1950, under the auspices of the American Mathematical Society. The Society planned originally to act as host for a Congress in 1940. Plans for the 1940 Congress were practically completed when the outbreak of World War II made it necessary for the Society to postpone the Congress to a more favorable date.

The 1950 Congress will be the third International Congress of Mathematicians to be held on the continent of North America. The first was held at Northwestern University in 1893 and the second at the University of Toronto in 1924. International Congresses were held at intervals of approximately four years, except when war intervened, until 1936. There has been no international gathering of mathematicians since that time, and it is the sincere hope of the Organizing Committee that the gathering in 1950 will be a truly international one, that American mathematicians will attend in large numbers, and that all other countries will be well represented. Mathematicians of all national and geographical groups are invited.

Place of Meeting. Harvard University will be the principal host institution for the Congress. A number of other institutions in metropolitan Boston will join in the entertainment of visitors to the Congress by arranging special features on their campuses. Mathematicians so desiring will be housed at a modest charge in the Harvard University dormitories and meals will be served in the University dining rooms. There will be accommodations for members of families, special provision being made for the care of children. Those who prefer to live in hotels can be comfortably provided for in Cambridge or Boston.

Organization. The Organizing Committee has invited more than a score of outstanding mathematicians to deliver stated addresses. There will be Conferences in four different fields; these are described more fully below. There will be seven Sections for the presentation of short contributed papers, and by invitation of the Section Chairmen there will be a small number of half-hour addresses as part of the Section programs.

In recent years mathematicians have been much impressed by the success of the conference method for presenting research in fields in which vigorous advances have just been made or are in progress. There will accordingly be a coordinated program of formal lectures and informal open discussion, and the stated addresses will be integrated as far as possible with the work of the Conferences. The following list of topics gives an indication of the nature of each of the Conferences:

ALGEBRA (Chairman, A. A. Albert): 1, Groups and universal algebra; 2, Structure theory of rings and algebras; 3, Arithmetic algebra; 4, Algebraic geometry.

ANALYSIS (Chairman, Marston Morse): 1, Algebraic tendencies in analysis; 2, Analysis and geometry in the large; 3, Extremal methods and geometric theory of functions of a complex variable.

APPLIED MATHEMATICS (Chairman, John von Neumann): 1, Partial differential equations; 2, Statistical mechanics; 3, Random processes in physics and communication.

TOPOLOGY, (Chairman, Hassler Whitney): 1, Homology and homotopy theory; 2, Fibre bundles and obstructions; 3, Differentiable manifolds; 4, Topological groups.

The Sections for the presentation of short contributed papers will be as follows: I, Algebra and Theory of Numbers; II, Analysis; III, Geometry and Topology; IV, Probability and Statistics, Actuarial Science, Economics; V, Mathematical Physics and Applied Mathematics; VI, Logic and Philosophy; VII, History and Education. Each member of the Congress may present only one contributed paper and the time allotted for each paper will be ten minutes. Abstracts for such papers should not exceed 400 words in length and must be submitted on blanks which may be secured from Professor J. R. Kline, the Secretary of the Congress, whose address is: University of Pennsylvania, Philadelphia 4, Pa. Abstracts must be in the hands of the Organizing Committee not later than May 15, 1950. The Organizing Committee has decided that contributed papers must be presented in person. The official languages of the Congress are English, French, German, Italian, and Russian.

Entertainment. There will be many interesting entertainment features, including a reception, a symphony concert, and a banquet. There will probably be a number of excursions.

Membership in the Congress. Membership in the Congress will be open to all qualified persons, whether they are able to be present in person or not. For regular members of the Congress the fee is $15.00; these persons will receive the Proceedings of the Congress. Members of families, accompanying Congress members and not participating in the scientific meetings, may become associate members, for whom the fee is $7.50; they will not present papers or receive the Proceedings, but will be entitled to many of the other privileges of membership. Only members and associate members of the Congress will have the privilege of residing in the Harvard dormitories.

Financial Support. Besides the support from Harvard University, generous subventions have been subscribed for the Congress by Bell Telephone Laboratories, Carnegie Corporation, General Electric Company, Institute for Advanced Study, Massachusetts Institute of Technology, National Research Council, Rockefeller Foundation, Standard Oil Development Company, and several private donors.

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The Mathematical Association of America will not hold its usual summer meeting in 1950, for it is believed that many members of the Association will desire to participate in the activities of the Congress. The ready opportunity of meeting a large number of distinguished foreign mathematicians and of hearing discussions on virtually every aspect of modern mathematics represents a climatic event in the history of American mathematics.

ON PAIRS OF TRIANGLES

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1. Introduction. In this MONTHLY [1] we have considered, particularly in connection with the orthopole, pairs of triangles inscribed in a circle. We also mentioned [2] that a theorem given by Murnaghan [3], Godeau [4], and Ramler [5] had been published before in a more general form by Kantor [6], and we further generalized this last theorem [2]. We will now consider orthopoles related to *any* pair of triangles placed in the same plane.

2. Angle associated with a pair of triangles. Let $A_1A_2A_3$ and $B_1B_2B_3$ be two triangles and denote by O_a and O_b, R_a and R_b the centers and the radii of their circumcircles.

Using complex coördinates, the base or unit circle being the circle $A_1A_2A_3$, let t_1, t_2, t_3 be the turns corresponding to A_1, A_2, A_3. Denote by

$$s_1 = t_1 + t_2 + t_3, \qquad s_2 = t_2t_3 + t_3t_1 + t_1t_2, \qquad s_3 = t_1t_2t_3$$

the symmetric functions of the t_i.

If c is the coördinate of O_b, the coördinate of B_i is $c+\lambda\tau_i$, λ being real and τ_1, τ_2, τ_3 being turns. We call

$$\sigma_1 = \tau_1 + \tau_2 + \tau_3, \qquad \sigma_2 = \tau_2\tau_3 + \tau_3\tau_1 + \tau_1\tau_2, \qquad \sigma_3 = \tau_1\tau_2\tau_3$$

the symmetric functions of the τ_i.

The conjugate to r being denoted by $\bar{r}$, the Simson line of the point t on the base circle as to $A_1A_2A_3$ has for its equation

$$tx - s_3\bar{x} = \frac{t^3 + s_1t^2 - s_2t - s_3}{2t}, \tag{1}$$

and that of the Simson line of the point $c+\lambda\tau$ as to $B_1B_2B_3$ is

$$\tau x - \sigma_3\bar{x} = \frac{\lambda\tau^3 + (2c + \lambda\sigma_1)\tau^2 - (2\bar{c}\sigma_3 + \lambda\sigma_2)\tau - \lambda\sigma_3}{2\tau}. \tag{2}$$

It follows that the points t and $c+\lambda\tau$ whose Simson lines are parallel to the diameter of the base circle passing through the unit point are s_3 and $c+\lambda\sigma_3$; if θ is such that

$$e^{i\theta} = \frac{\sigma_3}{s_3},$$

θ is the angle between the circumradii of $A_1A_2A_3$ and $B_1B_2B_3$ passing through the two points, respectively, under consideration.

Hence we derive easily from (1) and (2) the following theorem:

The points A and B of the circumcircles of the triangles $A_1A_2A_3$ and $B_1B_2B_3$, whose respective Simson lines as to those triangles are parallel to any given direction, are such that the circumradii, corresponding to those points, form a constant angle.

This angle plays an interesting part in the properties of a pair of triangles, and will be called the *angle associated with $A_1A_2A_3$ and $B_1B_2B_3$*.

3. Segment associated with a pair of triangles. If from any point as origin we draw two segments equal and parallel to O_aA and O_bB, the segment joining the extremities of these segments will be called the *segment associated with $A_1A_2A_3$ and $B_1B_2B_3$*; it will be denoted by l and

$$l^2 = \left(\frac{s_3}{\sigma_3} - \lambda\right)\left(\frac{\sigma_3}{s_3} - \lambda\right),$$

or, in terms of R_a, R_b and θ,

$$l^2 = R_a^2 + R_b^2 - 2R_aR_b\cos\theta.$$

4. Orthopolar ellipse of a pair of triangles. The equation to B_2B_3 being

$$x + \tau_2\tau_3\bar{x} = c + \tau_2\tau_3\bar{c} + \lambda(\tau_2 + \tau_3),$$

the coördinate of the orthopole of B_2B_3 as to $A_1A_2A_3$ is

$$\frac{\left[s_1 + c + \lambda\sigma_1 + \tau_1\left(\frac{s_3}{\sigma_3} - \lambda\right) + \bar{c}\,\frac{\sigma_3}{\tau_1}\right]}{2}.$$

It follows that the orthopoles of the sides of $B_1B_2B_3$ as to $A_1A_2A_3$ are on an ellipse E having for center the point

$$\frac{s_1 + c + \lambda\sigma_1}{2}$$

and for axes

$$(c\bar{c})^{1/2} \pm \left(\frac{s_3}{\sigma_3} - \lambda\right)^{1/2}\left(\frac{\sigma_3}{s_3} - \lambda\right)^{1/2},$$

these axes being parallel to

$$\frac{x}{\bar{x}} = = \frac{c^{1/2}\left(\frac{s_3}{\sigma_3} - \lambda\right)^{1/2}}{\bar{c}^{1/2}\left(\frac{\sigma_3}{s_3} - \lambda\right)^{1/2}}. \tag{3}$$

Similarly the orthopole of A_2A_3 as to $B_1B_2B_3$ is

$$\frac{s_1 + c + \lambda\sigma_1 + \left(\lambda\frac{\sigma_3}{s_3} - 1\right)t_1 - \bar{c}\frac{s_3}{t_1}}{2}$$

and it is easily found that orthopoles of the sides of $A_1A_2A_3$ as to $B_1B_2B_3$ are also on the ellipse E.

Hence we have the following theorem:

If $A_1A_2A_3$ and $B_1B_2B_3$ are two triangles whose circumcenters are at a distance d and if l is their associated segment, the orthopoles of the sides of each of the triangles as to the other triangle are on an ellipse having for center the midpoint of the segment joining the orthocenters of the triangles and having for axes $d \pm l$.

To find a construction for the direction of the axes, we consider, for a moment, that the unit-point is on O_aO_b, so that $\bar{c} = c$, and from (3) we then derive the following construction:

If A_0 and B_0 are the points on the circumcircles of $A_1A_2A_3$ and $B_1B_2B_3$ such that their respective Simson lines as to those triangles are parallel to O_aO_b, and if O_aB_0' is equal and parallel to O_bB_0, then the axes of E are parallel to the bisectors of the angle (O_aO_b, A_0B_0').

5. Kantor's theorem. When $O_a = O_b$ and $R_a = R_b$, we find that the six considered orthopoles are on a circle having for radius $R_a \cos \theta/2$; this is Kantor's theorem [6].

But the ellipse E becomes a circle when the circumcircles of the given triangles are only concentric.

When two triangles $A_1A_2A_3$ and $B_1B_2B_3$ are inscribed in two concentric circles, the orthopoles of the sides of each of the triangles as to the other are on a circle having for center the midpoint of the segment joining their orthocenters, and the square of the radius of that circle is

$$R_a^2 + R_b^2 - 2R_aR_b \cos \theta,$$

θ being the angle associated with the pair of triangles.

6. Orthopolar ellipse of a point as to a triangle. When $R_b = 0$, we find that the orthopoles of any three lines passing through a point O_b as to $A_1A_2A_3$ are on the ellipse having for center the midpoint of the segment joining O_b to the

orthocenter of $A_1A_2A_3$ and for axes $R_a \pm O_aO_b$; these axes are parallel to the bisectors of the angles between O_aO_b and the circumdiameter of $A_1A_2A_3$ passing through the point whose Simson line as to that triangle is parallel to O_aO_b.

This is the well known locus [7], [8] of the orthopole, as to $A_1A_2A_3$, of a variable straight line passing through the fixed point O_b.

7. Collinear orthopoles. In the special case when $d=l$, we have this theorem:

When two triangles $A_1A_2A_3$ and $B_1B_2B_3$ are so placed that the distance of their circumcenters equals their associated segment, the orthopoles of the sides of each of the triangles as to the other triangle are six collinear points.

8. Relation between triangle areas. This last theorem may also be derived as a special case from a property about triangle areas.

Call Δ_a and Δ_b the areas of $A_1A_2A_3$ and $B_1B_2B_3$ and Δ_α and Δ_β the areas of the triangles formed by the orthopoles of the sides of $B_1B_2B_3$ as to $A_1A_2A_3$ and by those of the sides of $A_1A_2A_3$ as to $B_1B_2B_3$. Then

$$16\sqrt{-1}\Delta_\alpha = -\left|\left(\frac{s_3}{\sigma_3}-\lambda\right)\tau_i + c\frac{\sigma_3}{\tau_i},\ \left(\frac{\sigma_3}{s_3}-\lambda\right)\tau_i + c\frac{\tau_i}{\sigma_3},\ 1\right|$$

$$= (d^2-l^2)\left|\tau_i,\ \frac{1}{\tau_i},\ 1\right| = -4\sqrt{-1}(d^2-l^2)\Delta_b.$$

Hence

$$\frac{\Delta_a}{\Delta_b} = -\frac{(d^2-l^2)}{4R_a^2}.$$

As we have a similar expression for Δ_β/Δ_a, we obtain the following theorem:

If Δ_a and Δ_b are the areas of two triangles $A_1A_2A_3$ and $B_1B_2B_3$ inscribed in circles having for radii R_a and R_b and if Δ_α and Δ_β are the areas of the triangles formed by the orthopoles of the sides of $B_1B_2B_3$ as to $A_1A_2A_3$ and by those of the sides of $A_1A_2A_3$ as to $B_1B_2B_3$, then

$$\frac{\Delta_a\Delta_\alpha}{\Delta_b\Delta_\beta} = \frac{R_b^2}{R_a^2}.$$

References

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2. R. Goormaghtigh, this MONTHLY, vol. 53, 1946, p. 524.
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4. R. Godeau, Mathesis, vol. 41, 1927, p. 72.
5. O. J. Ramler, this MONTHLY, vol. 47, 1940, p. 140.
6. S. Kantor, Mathesis, vol. 26, 1906, p. 144.
7. A. A. Krishnaswami Ayyangar, Educational Review, Madras, 1924.
8. J. Neuberg, Bulletin de l'Académie Royale de Belgique, juillet-août 1910.

SOME APPLICATIONS OF MATRICES IN THE THEORY OF EQUATIONS*

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1. The companion matrix. It is a very easy matter to find a solution of any polynomial equation. Thus I can immediately write out a solution of the cubic equation

$$x^3 - 9x^2 + 24x - 20 = 0,$$

namely, the matrix

$$A = \begin{bmatrix} 0 & 0 & 20 \\ 1 & 0 & -24 \\ 0 & 1 & 9 \end{bmatrix};$$

for it can very readily be shown that

$$A^3 - 9A^2 + 24A - 20I = 0.$$

We call A the companion matrix of the equation.

Of course such a solution is not of direct application in engineering practice, for it is the real or complex solutions that we ordinarily desire. The complex roots in the above case are all real, namely, 2, 2, 5. The question might naturally be asked if there is any relation between this matric solution and the complex solutions.

The answer is in the affirmative, for the companion matrix of every polynomial equation is similar to a triangular matrix having the complex roots in the principal diagonal. Thus for the example above

$$P^{-1}AP = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad P = \begin{bmatrix} 10 & 5 & 4 \\ -7 & -6 & -4 \\ 1 & 1 & 1 \end{bmatrix}.$$

The result is general. If the complex roots are distinct, the companion matrix is similar to a diagonal matrix whose elements are the complex roots. If there are multiple roots, these are linked with 1's. Thus the coefficients of an equation and its complex roots are related by the matric equation

$$P^{-1}AP = C$$

where A involves the coefficients and C the roots.

There is another well-known set of relations between the roots and the coefficients which is embodied in the theory of symmetric functions. This powerful method is sometimes tedious in its applications. It seems worth while to inquire if the companion matrix can be applied in any situations as a substitute for the theory of symmetric functions.

* Presented to the annual meeting of the Association in New York, December 30, 1949.

2. A theorem of Frobenius. A matrix

$$M = \begin{bmatrix} M_1 & O \\ O & M_2 \end{bmatrix}$$

where O denotes a rectangular block of 0's is called the direct sum of M_1 and M_2 and may be written

$$M = M_1 + M_2.$$

The companion matrix A of a polynomial $f(x)$ is similar to a direct sum of matrices

$$C = J_1 + J_2 + \cdots + J_k$$

where k is the number of distinct complex roots of $f(x) = 0$, and each matrix

$$J = \begin{bmatrix} r & 1 & 0 & 0 \cdots \\ 0 & r & 1 & 0 \cdots \\ 0 & 0 & r & 1 \cdots \\ 0 & 0 & 0 & r \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

has as many rows and columns as the multiplicity of the root r.

Now if $g(x)$ is any polynomial,

$$P^{-1}g(A)P = g(C) = g(J_1) + g(J_2) + \cdots + g(J_k)$$

where

$$g(J) = \begin{bmatrix} g(r) & g'(r) & \frac{1}{2}g''(r) & \frac{1}{3!}g'''(r) \cdots \\ 0 & g(r) & g'(r) & \frac{1}{2}g''(r) \cdots \\ 0 & 0 & g(r) & g'(r) \cdots \\ 0 & 0 & 0 & g(r) \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

It is now obvious that if $r_1, r_2, \cdots, r_k$ are the distinct roots of $f(x) = 0$, then $g(r_1), g(r_2), \cdots, g(r_k)$ are the characteristic roots of $g(C)$, which are in turn the characteristic roots of $g(A)$. If r_1 is a root of $f(x) = 0$ of multiplicity h, then $g(r_1)$ occurs at least h times among the roots of $g(A)$. Of course, r_1 and r_2 may be different and such that $g(r_1) = g(r_2)$, in which case $g(r_1)$ is of higher multiplicity than h.

This well-known result, which is due to Frobenius, is a very practicable one for the performance of Tschirnhausen transformations. For instance, let it be required to find the equation $F(x)=0$ whose roots are the squares of the roots of

$$f(x) = x^2 + x + 5 = 0.$$

The companion matrix is

$$A = \begin{bmatrix} 0 & -5 \\ 1 & -1 \end{bmatrix}.$$

In this case $g(x)=x^2$ so that the roots of

$$F(x) = |A^2 - xI| = \begin{vmatrix} -5-x & 5 \\ -1 & -4-x \end{vmatrix} = x^2 + 9x + 25 = 0$$

are the squares of the roots of $f(x)=0$.

In general the equation whose roots are the function $g(x)$ of the roots of $f(x)=0$ is

$$|g(A) - xI| = 0$$

where A is the companion matrix of $f(x)=0$.

3. The resultant. Now let $f(x)=0$ and $g(x)=0$ be two polynomial equations, and let A be the companion matrix and $r_1, r_2, \cdots, r_n$ the complex roots of $f(x)=0$. By the Frobenius theorem, the characteristic roots of $g(A)$ are $g(r_1)$, $g(r_2), \cdots, g(r_n)$. The product of these roots is the constant term of the characteristic equation of $g(A)$; that is,

$$(-1)^n |g(A)| = g(r_1)\cdot g(r_2)\cdot \cdots \cdot g(r_n).$$

Thus $g(A)$ is singular if and only if $f(x)=0$ and $g(x)=0$ have a root in common. In fact, $(-1)^n|g(A)|$ is the resultant of $f(x)=0$ and $g(x)=0$. This fact is well known.

The concepts of determinant and matrix have remained sadly tangled up to recent years, and even yet there is much popular confusion. Some results which originally were stated in the language of determinants have proved to be but special cases of more complete theorems concerning matrices. The above result is a case in point, for we can prove the following useful theorem which somehow seems to have been overlooked.

THEOREM 1. *Let A be the companion matrix of the polynomial $f(x)$. The nullity of the matrix $g(A)$ is equal to the number of complex roots which $f(x)=0$ and $g(x)=0$ have in common, a root being counted as many times as its multiplicity.*

Let $f(x)=0$ have a root r of multiplicity 4, for instance. Then $g(A)$ is similar to a direct sum of blocks, each corresponding to a different root. The block corresponding to the root r will have the form

$$g(J) = \begin{bmatrix} g(r) & g'(r) & \frac{1}{2}g''(r) & \frac{1}{3!}g'''(r) \\ 0 & g(r) & g'(r) & \frac{1}{2}g''(r) \\ 0 & 0 & g(r) & g'(r) \\ 0 & 0 & 0 & g(r) \end{bmatrix}.$$

If r is not a root of $g(x)=0$, then $g(J)$ is of nullity 0. If r is a simple root, $g(r)=0$, $g'(r)\neq 0$, and $g(J)$ is of nullity 1. If r is a double root, $g(r)=g'(r)=0$, $g''(r)\neq 0$, and $g(J)$ is of nullity 2, *etc.* Since the nullity of a direct sum is equal to the sum of the nullities of the component matrices, the theorem is now evident.

Thus, let

$$f(x) = x^3 + 3x^2 - 4, \qquad g(x) = x^3 + 6x^2 + 12x + 8.$$

We find

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix}, \qquad g(A) = \begin{bmatrix} 12 & 12 & 12 \\ 12 & 12 & 12 \\ 3 & 3 & 3 \end{bmatrix}.$$

Since $g(A)$ is of nullity 2, $f(x)=0$ and $g(x)=0$ have two roots in common. Actually in this case it is a common double root.

4. The equation of squared differences. There are many applications of the process described in the last paragraph, one of which is the calculation of the equation whose roots are the squared differences of the roots of the given equation.

Consider the equation $f(x)=0$ with roots $r_1, r_2, \cdots, r_n$ and consider an equation $g(x)=0$ to be found whose roots are the differences $y_k=r_i-r_j$. There are n^2 such differences, including n zeros, the remaining n^2-n differences occurring in opposite pairs. Since $y_k+r_j=r_i$ is a root of $f(x)=0$, it is true that

$$f(x) = 0, \qquad f(x+y) = 0$$

have a common root for every y satisfying $g(y)=0$. Thus $g(y)$ is the eliminant of these equations. If A is the companion matrix of $f(x)=0$, then

$$|f(A + yI)| = g(y) = y^n F(y^2).$$

The roots of $F(z)=0$, then, are the square differences

$$(r_i - r_j)^2, \qquad i > j,$$

and the constant term of $F(z)$ with proper sign is the discriminant of $f(x)=0$.

Thus for

$$f(x) = x^3 - 7x + 6^*$$

$$A = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & 7 \\ 0 & 1 & 0 \end{bmatrix}, \qquad A + yI = \begin{bmatrix} y & 0 & -6 \\ 1 & y & 7 \\ 0 & 1 & y \end{bmatrix},$$

$$f(A + yI) = \begin{bmatrix} y^3 - 7y & -18y & -18y^2 \\ 3y^2 & y^3 + 14y & 21y^2 - 18y \\ 3y & 3y^2 & y^3 + 14y \end{bmatrix},$$

$$|f(A + yI)| = y^3(y^6 - 42y^4 + 441y^2 - 400),$$

so that

$$F(z) = z^3 - 42z^2 + 441z - 400.$$

The discriminant is 400.

5. The discriminant. The above method is not the most direct way of finding the discriminant of an equation, however, since the latter is the eliminant of $f(x) = 0$ and $f'(x) = 0$. That is,

$$\Delta = (-1)^n |f'(A)|$$

where A is the companion matrix of $f(x) = 0$.

But an even more valuable theorem than this can be deduced:

THEOREM 2. *If A is the companion matrix of $f(x)$, then the rank of $f'(A)$ is the number of distinct complex roots of the equation $f(x) = 0$.*

Suppose that $f(x)$ has h_1 simple roots, h_2 double roots, h_3 triple roots, *etc.* Then $f(x) = 0$ and $f'(x) = 0$ have in common h_2 simple roots, h_3 double roots, *etc.*, so that by Theorem 1 the nullity of $f'(A)$ is

$$\nu = h_2 + 2h_3 + 3h_4 + \cdots.$$

But the order of A is

$$n = h_1 + 2h_2 + 3h_3 + 4h_4 + \cdots$$

so that the rank of $f'(A)$ is

$$r = n - \nu = h_1 + h_2 + h_3 + h_4 + \cdots,$$

and the number of distinct roots of $f(x) = 0$.

Let us consider the general cubic equation

$$f(x) = x^3 + bx^2 + cx + d = 0.$$

Then

* Burnside and Panton, The theory of equations, Dublin University Press, 1912, v. 1, p. 83.

$$f'(A) = \begin{bmatrix} c & -3d & bd \\ 2b & -2c & bc - 3d \\ 3 & -b & b^2 - 2c \end{bmatrix}$$

whose determinant multiplied by $(-1)^3$ is

$$b^2c^2 - 4c^3 - 4b^3d - 27d^2 + 18bcd,$$

the discriminant.

If we set

$$f(x) = x^3 - 3rx^2 + 3r^2x - r^3 = (x - r)^3,$$

then

$$f'(A) = \begin{bmatrix} 3r^2 & 3r^3 & 3r^4 \\ -6r & -6r^2 & -6r^3 \\ 3 & 3r & 3r^2 \end{bmatrix}$$

which is of rank 1.

For every $f(x)$ of degree n, $f'(A) \neq 0$ since $f(x)$ is the minimum function of A. This does not constitute a proof of the fundamental theorem of algebra.

6. An illustration. I am tempted to make one further application of this method, even though the problem to which it is applied is rather trivial. Let

$$f(x) = f_1(x) \cdot f_2(x)$$

where $f_1(x)$ and $f_2(x)$ are relatively prime. Then the companion matrix A of $f(x)$ is similar to

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where A_1 is the companion matrix of $f_1(x)$, and A_2 is the companion matrix of $f_2(x)$. Then

$$f'(x) = f_1(x) \cdot f_2'(x) + f_2(x) \cdot f_1'(x)$$

and, since $f_1(A_1) = f_2(A_2) = 0$,

$$f'(A) \overset{s}{=} f'(A_1 + A_2) = \begin{bmatrix} f_2(A_1) \cdot f_1'(A_1) & 0 \\ 0 & f_1(A_2) \cdot f_2'(A_2) \end{bmatrix}.$$

Hence the discriminant of $f(x) = 0$ is

$$\pm |f'(A)| = \pm |f_1(A_2)| \cdot |f_2(A_1)| \cdot |f_1'(A_1)| \cdot |f_2'(A_2)|$$

which is $\pm$ the square of the resultant of $f_1(x)$ and $f_2(x)$ multiplied by the product of their discriminants.

7. Symmetric functions. Since we have been able to use matric methods in place of symmetric function theory so effectively in special instance, it would be interesting to see if the general theory of symmetric functions can be so developed.

It is well known and easily proved that every polynomial which is symmetric in the roots $r_1, r_2, \cdots, r_n$ of an equation $f(x)=0$ is equal to a sum of symmetric polynomials of the type

$$\sum r_1^{e_1} r_2^{e_2} \cdots r_k^{e_k}, \qquad e_1 \geqq e_2 \geqq \cdots \geqq e_k.$$

This polynomial* is the coefficient of $\lambda_{e_1}\lambda_{e_2} \cdots \lambda_{e_k}$ in the expansion of

$$\prod_{i=1}^{n}(1 + \lambda_1 r_i + \lambda_2 r_i^2 + \cdots + \lambda_{e_1} r_i^{e_1})$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are indeterminates.

Let

$$f(x) = x^n + ax^{n-1} + bx^{n-2} + \cdots = 0$$

have the roots $r_1, r_2, \cdots, r_n$, and let A be its companion matrix. Form the polynomial

$$g(x) = 1 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_{e_1} x^{e_1}.$$

Then $|g(A)|$ is the constant term of the equation whose roots are the functions $g(x)$ of the roots of $f(x)=0$, so that

$$(-1)^n |g(A)| = \prod_{i=1}^{n} (1 + \lambda_1 r_i + \lambda_2 r_i^2 + \cdots + \lambda_{e_1} r_i^{e_1}).$$

Upon expanding both sides and equating coefficients of like powers of the λ's we obtain the symmetric functions of the roots of $f(x)=0$ expressed as polynomials in the coefficients of $f(x)$. Thus we have a short proof of the Fundamental Theorem on Symmetric Functions. Let the leading coefficient of $f(x)$ be 1. Every polynomial in the roots of $f(x)=0$ with coefficients in any field which is symmetric in these roots is expressible as a polynomial in the coefficients of $f(x)$ with coefficients in that field.

Moreover, the method is practicable and can be used to compute one or all of the symmetric polynomials in n roots of degree $\leqq k$ without recursive relations. Thus, suppose that $n=3$ and that

$$f(x) = x^3 + bx^2 + cx + d.$$

Let it be required to find $\sum r_1^3 r_2^2$. We have

* M. Faber, Archiv der Mathematik und Physik, 16 (1910), 144.

$$A=\begin{bmatrix}0 & 0 & -d\\ 1 & 0 & -c\\ 0 & 1 & -b\end{bmatrix}, \qquad g(x)=1+\lambda_1 x+\lambda_2 x^2+\lambda_3 x^3,$$

$$g(A)=\begin{bmatrix}-\lambda_3 d+1 & \lambda_3 bd-\lambda_2 d & \lambda_3(cd-b^2d)+\lambda_2 bd-\lambda_1 d\\ -\lambda_3 c+\lambda_1 & \lambda_3(bc-d)-\lambda_2 c+1 & \lambda_3(c^2-b^2c+bd)+\lambda_2(bc-d)-\lambda_1 c\\ -\lambda_3 b+\lambda_2 & \lambda_3(b^2-c)-\lambda_2 b+\lambda_1 & \lambda_3(-d+2bc-b^3)+\lambda_2(b^2-c)-\lambda_1 b+1\end{bmatrix}.$$

Each element of $g(A)$ is linear in 1, λ_1, λ_2, λ_3, so that this determinant can be written as a sum of 64 determinants each of which is equal to a monomial in the λ's. Thus to calculate $\sum r_1^3 r_2^2$ we need the sum of those determinants which are multiples of $\lambda_3\lambda_2$, namely,

$$\begin{vmatrix}-d & -d & 0\\ -c & -c & 0\\ -b & -b & 1\end{vmatrix}+\begin{vmatrix}-d & 0 & bd\\ -c & 1 & bc-d\\ -b & 0 & b^2-c\end{vmatrix}+\begin{vmatrix}0 & bd & 0\\ 0 & bc-d & 0\\ 1 & b^2-c & 1\end{vmatrix}$$

$$+\begin{vmatrix}0 & 0 & cd-b^2d\\ 0 & 1 & c^2-b^2c+bd\\ 1 & 0 & -d+2bc-b^3\end{vmatrix}+\begin{vmatrix}1 & bd & bd\\ 0 & bc-d & bc-d\\ 0 & b^2-c & b^2-c\end{vmatrix}$$

$$+\begin{vmatrix}1 & -d & cd-b^2d\\ 0 & -c & c^2-b^2c+bd\\ 0 & -b & -d+2bc-b^3\end{vmatrix}=2b^2d+cd-bc^2.$$

Similarly

$$\sum r_1^3 r_2 = b^2c-bd-2c^2$$

is the coefficient of $\lambda_3\lambda_1$ in the expansion of $|g(A)|$. All other symmetric polynomials may be calculated similarly.

THE VECTOR POTENTIAL OF A SOLENOIDAL VECTOR

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If the vector $\mathbf{f}(\mathbf{r})$ is solenoidal in a region R (div $\mathbf{f}=0$), $\mathbf{f}$ can be expressed as the rotation of a vector $\mathbf{g}$, its vector potential; that is,

$$\mathbf{f} = \text{rot}\,\mathbf{g}.$$

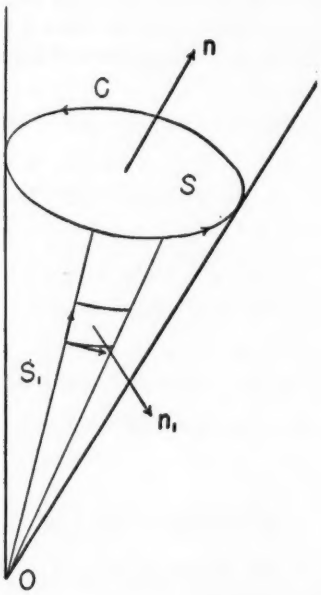
The function $\mathbf{g}$ is arbitrary to an additive gradient of a scalar. The following method of finding $\mathbf{g}$ is independent of the coordinate system and involves only

simple integration.

The integral of $\mathbf{f}$ over a surface S bounded by a closed curve C, $\mathbf{r}=\mathbf{r}(s)$, is

$$\int_S \mathbf{n}\cdot\mathbf{f}dS = \int_S \mathbf{n}\cdot\operatorname{rot}\mathbf{g}dS = \oint_C \tau\cdot\mathbf{g}ds \tag{1}$$

by Stokes' Theorem. Let S_1 be the surface of a cone with vertex at the origin and having C as directrix and boundary; note the figure. The parametric equa-



tion of S_1 is $\mathbf{r}_1=\lambda\mathbf{r}(s)$, where s is the length of arc on C measured from some fixed point of the curve. Hence the vector element of area on S_1 is

$$\mathbf{n}_1 dS_1 = \frac{\partial\mathbf{r}_1}{\partial s}\times\frac{\partial\mathbf{r}_1}{\partial\lambda}dsd\lambda = \tau(s)\times\mathbf{r}(s)\lambda dsd\lambda,$$

where τ is a unit tangent vector along C. Hence the surface integral of $\mathbf{f}$ over S_1 is

$$\int_{S_1}\mathbf{n}_1\cdot\mathbf{f}(\mathbf{r}_1)dS_1 = \oint_C \tau\cdot\left\{\int_0^1 \mathbf{r}(s)\times\mathbf{f}(\lambda\mathbf{r})\lambda d\lambda\right\}ds. \tag{2}$$

The unit normals $\mathbf{n}$ and $\mathbf{n}_1$ are both external to the region enclosed by S and S_1. Since div $\mathbf{f}=0$, the sum of the surface integrals (1) and (2) is zero; hence for any closed curve C in R (compatible with the conditions of Stokes' Theorem)

$$\oint_C \tau \cdot \left\{ \mathbf{g} + \int_0^1 \mathbf{r}(s) \times \mathbf{f}(\lambda \mathbf{r}) \lambda d\lambda \right\} ds = 0.$$

The vector in braces is therefore irrotational and may be expressed as the gradient of a scalar. We thus obtain the particular solution

$$\mathbf{g}(\mathbf{r}) = -\mathbf{r} \times \int_0^1 \mathbf{f}(\lambda \mathbf{r}) \lambda d\lambda, \tag{3}$$

provided $\mathbf{r} \times \mathbf{f}(\lambda \mathbf{r}) \neq 0$ and the integral converges in case it is improper. When $\lambda \mathbf{f}(\lambda \mathbf{r}) \to \infty$ as $\lambda \to 0$ we assume that

$$\lim_{\lambda \to 0} \lambda^p \mathbf{f}(\lambda \mathbf{r}) = \mathbf{A}, \qquad 1 < p < 2, \tag{4}$$

in order to ensure convergence.

The formula (3) is due to H. Liebmann* and has remained unnoticed in the literature. Liebmann apparently regarded his derivation of (3) as heuristic in character, and proceeded to verify that rot $\mathbf{g} = \mathbf{f}$, using rectangular coordinates for the computation. Since his proof is rather long we give a brief vector-analytic verification. We assume that $\nabla \mathbf{f}(\lambda \mathbf{r})$ and $d\mathbf{f}(\lambda \mathbf{r})/d\lambda$ are continuous in the region R and the interval $0 < \lambda \leqq 1$; then from (3)

$$\text{rot } \mathbf{g} = \int_0^1 \text{rot } \{\mathbf{f}(\lambda \mathbf{r}) \times \mathbf{r}\} \lambda d\lambda. \tag{5}$$

We now apply the formula

$$\text{rot } (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{u} \text{ div } \mathbf{v} - \mathbf{v} \text{ div } \mathbf{u}$$

to the integrand; putting $\mathbf{r}_1 = \lambda \mathbf{r}$,

$$\text{rot } \{\mathbf{f}(\mathbf{r}_1) \times \mathbf{r}\} = \mathbf{r} \cdot \nabla \mathbf{f}(\mathbf{r}_1) - \mathbf{f}(\mathbf{r}_1) \cdot \nabla \mathbf{r} + \mathbf{f}(\mathbf{r}_1) \text{ div } \mathbf{r} - \mathbf{r} \text{ div } \mathbf{f}(\mathbf{r}_1),$$

where the differential operations ∇, rot, div are all with respect to the vector $\mathbf{r}$. Now if ∇_1 and div_1 refer to the vector $\mathbf{r}_1$, we have

$$\nabla \mathbf{f}(\mathbf{r}_1) = \nabla \mathbf{r}_1 \cdot \nabla_1 \mathbf{f}(\mathbf{r}_1) = \lambda \mathbf{I} \cdot \nabla_1 \mathbf{f}(\mathbf{r}_1) = \lambda \nabla_1 \mathbf{f}(\mathbf{r}_1),$$
$$\text{div } \mathbf{f}(\mathbf{r}_1) = \lambda \text{ div}_1 \mathbf{f}(\mathbf{r}_1) = 0;$$

and since $\nabla \mathbf{r} = \mathbf{I}$, div $\mathbf{r} = 3$,

$$\text{rot } \{\mathbf{f}(\mathbf{r}_1) \times \mathbf{r}\} = \lambda \mathbf{r} \cdot \nabla_1 \mathbf{f}(\mathbf{r}_1) + 2\mathbf{f}(\mathbf{r}_1).$$

Now

$$\frac{d}{d\lambda} \mathbf{f}(\mathbf{r}_1) = \frac{d\mathbf{r}_1}{d\lambda} \cdot \nabla_1 \mathbf{f}(\mathbf{r}_1) = \mathbf{r} \cdot \nabla_1 \mathbf{f}(\mathbf{r}_1),$$

* Leipziger Berichte, Bd. 60 (1908), pp. 176–189.

$$\lambda \operatorname{rot} \{\mathbf{f}(\mathbf{r}_1) \times \mathbf{r}\} = \lambda^2 \frac{d}{d\lambda} \mathbf{f}(\mathbf{r}_1) + 2\lambda f(\mathbf{r}_1) = \frac{d}{d\lambda} \{\lambda^2 \mathbf{f}(\mathbf{r}_1)\};$$

and since this expression is the integrand in (5),

$$\operatorname{rot} \mathbf{g} = \lambda^2 \mathbf{f}(\mathbf{r}_1) \Big|_{\lambda=0}^{\lambda=1} = \mathbf{f}(\mathbf{r}).$$

When the integral is improper the lower limit is still zero, for, from (4),

$$\lim_{\lambda \to 0} \lambda^2 \mathbf{f}(\mathbf{r}_1) = \lim_{\lambda \to 0} \mathbf{A} \lambda^{2-p} = 0, \qquad 1 < p < 2.$$

Thus under the conditions stated, rot $\mathbf{g}(\mathbf{r}) = \mathbf{f}(\mathbf{r})$.

When $\mathbf{r} \times \mathbf{f}(\lambda \mathbf{r}) = 0$ or when the integral in (3) diverges Liebmann's formula is inapplicable. We proceed to extend the method above and find $\mathbf{g}$ in these cases of failure.

If $\mathbf{r} \times \mathbf{f}(\lambda \mathbf{r}) \neq 0$, but the integral in (3) is a divergent improper integral, we may replace it by

$$\mathbf{g}(\mathbf{r}) = \mathbf{r} \times \int_1^\infty \mathbf{f}(\lambda \mathbf{r}) \lambda d\lambda, \tag{6}$$

provided that this integral converges. The convergence will be assured if we assume that

$$\lim_{\lambda \to \infty} \lambda^p \mathbf{f}(\lambda \mathbf{r}) = \mathbf{A}, \qquad p > 2. \tag{7}$$

For the preceding argument then shows that when $\mathbf{g}$ is given by (6),

$$\operatorname{rot} \mathbf{g} = -\lambda^2 \mathbf{f}(\lambda \mathbf{r}) \Big|_{\lambda=1}^{\lambda=\infty} = \mathbf{f}(\mathbf{r})$$

since from (7)

$$\lim_{\lambda \to \infty} \lambda^2 \mathbf{f}(\lambda \mathbf{r}) = \lim_{\lambda \to \infty} \frac{\mathbf{A}}{\lambda^{p-2}} = 0, \qquad p > 2 \tag{8}$$

Formula (7) was obtained by integrating over the cone

$$\mathbf{r}_1 = \lambda \mathbf{r}(s), \qquad 1 \leqq \lambda < \infty,$$

extending from the curve C to infinity. The surface integral over the portion of an infinite sphere about O intercepted by the cone vanishes in view of (8).

If $\mathbf{f}(\mathbf{r})$ is homogeneous of degree n, namely,

$$\mathbf{f}(\lambda \mathbf{r}) = \lambda^n \mathbf{f}(\mathbf{r}), \tag{9}$$

and $n \neq -2$, both (3) and (6) give

$$\mathbf{g}(\mathbf{r}) = \frac{\mathbf{f}(\mathbf{r}) \times \mathbf{r}}{n+2}. \tag{10}$$

When $n \geqq -1$, the integral in (3) is proper; when $-2 < n < -1$, (4) is satisfied with $p = -n$; and when $n < -2$, (7) is satisfied with $p = -n$. Moreover we may verify directly that rot $\mathbf{g} = \mathbf{f}$ if we make use of the extension of Euler's formula

$$\mathbf{r} \cdot \nabla \mathbf{f}(\mathbf{r}) = n\mathbf{f}(\mathbf{r}) \tag{11}$$

obtained by differentiating (9) with respect to λ and then putting $\lambda = 1$. In fact we have from (10)

$$\text{rot } \mathbf{g} = \frac{1}{n+2} \{\mathbf{r} \cdot \nabla \mathbf{f} - \mathbf{f} \cdot \nabla \mathbf{r} + \mathbf{f} \text{ div } \mathbf{r}\} = \mathbf{f}.$$

Neither (3) or (6) apply when $n = -2$. In this case we may use the formula

$$\mathbf{g} = \mathbf{e} \times \int_0^\infty \mathbf{f}(\mathbf{r} + \lambda \mathbf{e}) d\lambda \tag{12}$$

where $\mathbf{e}$ is a constant unit vector of arbitrary direction. Indeed equation (12) applies generally whenever the integral converges, for example, when

$$\lim_{\lambda \to \infty} \lambda^p \mathbf{f}(\mathbf{r} + \lambda \mathbf{e}) = \mathbf{A}, \qquad p > 1. \tag{13}$$

For if we put $\mathbf{r}_1 = \mathbf{r} + \lambda \mathbf{e}$ and use the formula for rot $(\mathbf{u} \times \mathbf{v})$ cited above

$$\text{rot } \mathbf{g} = -\mathbf{e} \cdot \int_0^\infty \nabla \mathbf{f}(\mathbf{r}_1) d\lambda = -\mathbf{e} \cdot \int_0^\infty \nabla_1 \mathbf{f}(\mathbf{r}_1) d\lambda,$$

since

$$\nabla f(\mathbf{r}_1) = \nabla \mathbf{r}_1 \cdot \nabla_1 \mathbf{f}(\mathbf{r}_1) = \nabla \mathbf{r} \cdot \nabla_1 \mathbf{f}(\mathbf{r}_1) = \nabla_1 \mathbf{f}(\mathbf{r}_1),$$
$$\text{div } \mathbf{f}(\mathbf{r}_1) = \text{div}_1 \mathbf{f}(\mathbf{r}_1) = 0.$$

Now

$$\frac{d}{d\lambda} \mathbf{f}(\mathbf{r}_1) = \frac{d\mathbf{r}_1}{d\lambda} \cdot \nabla_1 \mathbf{f}(\mathbf{r}_1) = \mathbf{e} \cdot \nabla_1 \mathbf{f}(\mathbf{r}_1)$$

and hence

$$\text{rot } \mathbf{g} = -\mathbf{f}(\mathbf{r}_1) \Big|_{\lambda=0}^{\lambda=\infty} = \mathbf{f}(\mathbf{r}),$$

for at the upper limit

$$\lim_{\lambda \to \infty} \mathbf{f}(\mathbf{r}_1) = \lim_{\lambda \to \infty} \frac{\mathbf{A}}{\lambda^p} = 0, \qquad p > 1, \tag{14}$$

from (13)

The integral (10) is obtained by integrating over the cylinder

$$\mathbf{r}_1 = \mathbf{r}(s) + \lambda\mathbf{e}. \qquad 0 \leqq \lambda < \infty,$$

extending from the curve C to infinity. Its surface element is

$$\mathbf{n}_1 dS_1 = \frac{\partial \mathbf{r}_1}{\partial s} \times \frac{\partial \mathbf{r}_1}{\partial \lambda}\, ds d\lambda = \boldsymbol{\tau} \times \mathbf{e} ds d\lambda,$$

where $\mathbf{n}_1$ is the external unit normal, and

$$\int_{S_1} \mathbf{n}_1 \cdot \mathbf{f}(\mathbf{r}_1) dS_1 = \oint_C \boldsymbol{\tau} \cdot \left\{ \int_0^\infty \mathbf{e} \times \mathbf{f}(\mathbf{r}_1) d\lambda \right\} ds. \tag{15}$$

The cylinder is closed by the cap S (over which $\mathbf{n}$ is now the internal normal) and by a portion of an infinite sphere about O. The surface integral over the latter vanishes in view of (14). Hence the difference of the surface integrals in (1) and (15) vanishes, and we obtain the particular value of $\mathbf{g}$ in (12).

By way of summary we state the

THEOREM. *If div* $\mathbf{f} = 0$ *the vector potential* $\mathbf{g}$ *of* $\mathbf{f}$ *is given by any one of the vector functions*

$$-\mathbf{r} \times \int_0^1 \mathbf{f}(\lambda\mathbf{r}) \lambda d\lambda, \qquad \mathbf{r} \times \int_1^\infty \mathbf{f}(\lambda\mathbf{r}) \lambda d\lambda, \qquad \mathbf{e} \times \int_0^\infty \mathbf{f}(\mathbf{r} + \lambda\mathbf{e}) d\lambda.$$

for which the integral converges. In particular if $\mathbf{f}$ *is homogeneous of degree* $n = -2$,

$$\mathbf{g} = \frac{\mathbf{f}(\mathbf{r}) \times \mathbf{r}}{n + 2}.$$

Example 1. The vector $\mathbf{f} = \mathbf{a}$ (constant) is solenoidal. Since $\mathbf{f}(\lambda\mathbf{r}) = \mathbf{f}(\mathbf{r})$ we apply (10) with $n = 0$: thus $\mathbf{g} = \mathbf{a} \times \mathbf{r}/2$.

Example 2. The vector $\mathbf{f} = \mathbf{r}/r^3 = -\nabla_{\mathbf{r}}^1$ is solenoidal and $\mathbf{f}(\lambda\mathbf{r}) = \lambda^{-2}\mathbf{f}(\mathbf{r})$. Hence formula (10) is inapplicable, but we may use (12) since

$$\lim_{\lambda \to \infty} \lambda^2 \frac{\mathbf{r} + \lambda\mathbf{e}}{|\mathbf{r} + \lambda\mathbf{e}|^3} = \mathbf{e}.$$

If we write $\mathbf{r} \cdot \mathbf{e} = r \cos\theta$,

$$(\mathbf{r} + \lambda\mathbf{e}) \cdot (\mathbf{r} + \lambda\mathbf{e}) = r^2 + 2\lambda r \cos\theta + \lambda^2$$

and hence

$$\mathbf{g} = \mathbf{e} \times \mathbf{r} \int_0^\infty \frac{d\lambda}{(r^2 + 2\lambda r \cos\theta + \lambda^2)^{3/2}}.$$

Making the substitution

$$t = \frac{\lambda + r\cos\theta}{r\sin\theta},$$

we find that

$$\mathbf{g} = \frac{\mathbf{e}\times\mathbf{r}}{r^2\sin^2\theta}\int_{\cot\theta}^{\infty}\frac{dt}{(1+t^2)^{3/2}} = \frac{\mathbf{e}\times\mathbf{r}}{r^2\sin^2\theta}(1-\cos\theta) = \frac{\mathbf{e}\times\mathbf{r}}{r^2(1+\cos\theta)}.$$

If we choose $\mathbf{e}=\mathbf{k}$ and write $\mathbf{e}\times\mathbf{r}=r\sin\theta\,\mathbf{c}$, where $\mathbf{c}$ is a unit vector, we have also

$$\mathbf{g} = (1-\cos\theta)\frac{\mathbf{c}}{r\sin\theta} = (1-\cos\theta)\nabla\phi,$$

where r, θ, ϕ are spherical coordinates.

This leads to an expression for the solid angle Ω_0 subtended at O by a closed curve C; for if S is a surface spanned over C

$$\Omega_0 = \int_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,dS = \oint_C \mathbf{r}\cdot\mathbf{g}\,ds = \oint_C (1-\cos\theta)\,d\phi.$$

Query: We may take $\mathbf{g}=(C-\cos\theta)\nabla\phi$, since $\mathbf{g}$ is arbitrary to an additive gradient. If we now compute the solid angle subtended by the circle $\theta=$const. on a sphere, $\Omega_0=(C-\cos\theta)2\phi$; how can this be true for any value of the constant C? Discuss the formula $\Omega_0=\oint(C-\cos\theta)d\phi$ for a spherical curve that does not enclose either pole.

Example 3. The vector $\mathbf{f}=r^n\mathbf{e}\times\mathbf{r}$ is solenoidal as we see on writing it

$$\mathbf{f} = \frac{1}{n+2}\,\mathbf{e}\times\nabla r^{n+2}.$$

Moreover, $\mathbf{f}(\lambda\mathbf{r})=\lambda^{n+1}\mathbf{f}(\mathbf{r})$; hence if $n\neq -3$ we may apply (10) and obtain

$$\mathbf{g} = \frac{r^n(\mathbf{e}\times\mathbf{r})\times\mathbf{r}}{n+3}.$$

When $n=-3$ this formula fails. We then use (12) to obtain

$$\mathbf{g} = \mathbf{e}\times(\mathbf{e}\times\mathbf{r})\int_0^{\infty}\frac{d\lambda}{|\mathbf{r}+\lambda\mathbf{e}|^3} = \frac{\mathbf{e}\times(\mathbf{e}\times\mathbf{r})}{r^2(1+\cos\theta)}$$

on using the value of the integral obtained in Example 2.

MATHEMATICAL NOTES

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ENVELOPES OF PLANE CURVES

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As we know, the envelope of a one parameter family of plane curves touches each member at the point they have in common. The usual proof of this result is analytic, that is, by calculating the slopes of the two tangent lines (the writer has seen no other proof); but it is well to give a geometric proof if only for its perspicuity.

As our definition of the envelope we take the usual one, namely that it is the locus of the points of intersection of "consecutive" members of the family.

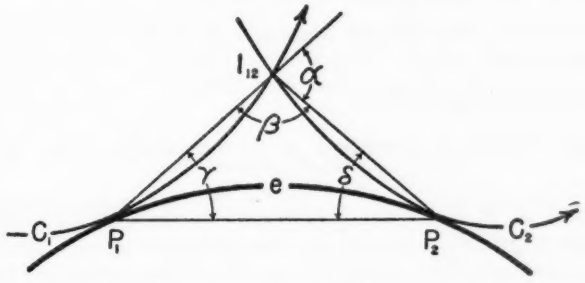
We let e denote the envelope of the family $\{f(x, y, c)=0\}$. We refer to the members $f(x, y, c_1)=0$ and $f(x, y, c_j)=0$ as the curves c_1 and c_j; symbols like these will sometimes represent values of the parameter but more often the curves associated with such values. We let P_j denote the point in which c_j and e meet, I_{jk} that in which c_j and c_k meet. I_{jk} and I_{kj} are, of course, the same point; and assuming the necessary continuity, we see that P_j can be represented equally well by I_{jj}. We shall be dealing with local properties and take c_1, c_2, c_3, c_j, c_k so that they differ from each other by sufficiently little. It is necessary for us to make three basic assumptions about the family $\{f(x, y, c)=0\}$, all geometric and quite natural. First we assume that P_2 lies between P_1 and P_3 on e if $c_1<c_2<c_3$; in other words that, locally at least, there is a direction of advance for the members of the family along the envelope. Second, that the point I_{2k} lies between I_{1k} and I_{3k} on c_k if $c_1<c_2<c_3$; this is simply a way of saying that the approach of I_{jk} along c_k to its limiting position $I_{kk}\equiv P_k$ is to be continuous. Third, that only one branch of c_j passes through P_j, that is, that P_j is not a multiple point with respect to c_j. This last excludes the nodal locus, ordinarily not a part of the envelope, which even analytically complicates the operation of finding the envelope.

Only one branch of c_j passes through P_j, and so we can designate one end (symbolized by c_j+) of the curve as the $+$ or forward end with respect to P_j, the other (symbolized by c_j-) as the $-$ or rearward end. We achieve a certain uniformity by taking this forward end in the general direction of advance along e for every local curve c_j.

We now let c_k approach c_2 and get a limiting form of the relationship conveyed by the second assumption, namely that $I_{22}\equiv P_2$ lies between I_{21} and I_{23} on c_2 if $c_1<c_2<c_3$. That is, I_{21} and I_{23} lie on opposite ends of the curve c_2 emanating from P_2. That is, if c_2- (alternatively c_2+) meets c_1, then c_2+ (alternatively c_2-) meets c_3. But c_2 is a general member of the family and its behavior vis-à-vis curves with larger and smaller parameter values must characterize all local

members. So it is c_1+ (alternatively c_1-) that meets c_2 or, taking into account the ends as well as the curve, it is c_1+ (alternately c_1-) that meets c_2- (alternatively c_2+). In short, c_1 and c_2 meet locally in unlike ends.

Taking the direction of advance along e to the right in the figure, we assume that it is the rear $(-)$ end of c_2 that meets the forward $(+)$ end of c_1—this is no real restriction—and represent the two curves as in the diagram. We draw the lines P_1P_2, P_1I_{12}, and $I_{12}P_2$ and consider the triangle $P_1I_{12}P_2$.



As the curve c_2 moves toward coincidence with c_1, I_{12} approaches P_1 on the one hand and P_2 on the other (definition of envelope), the directed lines $\overrightarrow{P_1I_{12}}$ and $\overrightarrow{I_{12}P_2}$ move into the positions of the forward tangents to these curves at the same point I_{12}, the angle α between the two directed lines approaches zero, β the supplement of α approaches π, and the angles γ and δ individually approach zero. Simultaneously, P_2 approaches P_1 along e and P_1P_2 swings into the position of the tangent to e at P_1, while P_1I_{12} becomes the tangent to c_1 at P_1, so that c_1 and e touch each other at P_1.

The fact that c_1 and c_2 meet in unlike ends is essential to the proof. Otherwise, $\overrightarrow{P_1I_{12}}$ and $\overrightarrow{I_{12}P_2}$ would in the limit be oppositely directed rays of the same line (the tangent to c_1 at I_{12}), α would approach π instead of zero, β would approach zero, $\gamma+\delta$ would approach π, so that γ would be indeterminate and our proof would fail.

This fact actually characterizes the envelope. That is, if we take any other curve g passing through P_1 and making a nonzero angle with e, defining the rear and forward directions of the local members of the family with respect to g, then c_1 and c_2 will meet in like ends. We can see this by appealing to a diagram or, more abstractly, by remembering that otherwise, using the very same reasoning that was used for e, we could prove that g and c_1 touch each other.

ON THE ORTHOCENTROIDAL CIRCLE

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Let H, G, O be the orthocenter, centroid, and circumcenter of triangle ABC. Let the feet of the altitudes be denoted by A_h, B_h, C_h, and the midpoints of the

sides by A_m, B_m, C_m.

If G_a, G_b, G_c are the centroids of triangles HBC, HCA, HAB, then triangle $G_aG_bG_c$ is homothetic to triangle $A_mB_mC_m$ under the homothety $(H, 2/3)$. The lines GG_a, GG_b, GG_c are thus parallel, respectively, to OA_m, OB_m, OC_m, and:

1. *Point G is the orthocenter of triangle $G_aG_bG_c$.*

It follows that $GG_a=(2/3)OA_m=(1/3)AH$. Therefore, if N is the intersection of AG_a and GH, $GN=(1/3)NH$, and we have:

2. *The lines AG_a, BG_b, CG_c are concurrent at N, the center of the nine point circle of triangle ABC.*

Also, since $NG_a=-(1/3)NA$:

3. *Triangle $G_aG_bG_c$ is homothetic to triangle ABC under the homothety $(N, -1/3)$.*

The vertices A, B, C are the orthocenters of triangles HBC, HCA, HAB, and the orthocentroidal circles [1] of these triangles are the circles on AG_a, BG_b, CG_c as diameters. The centers O_a, O_b, O_c of these circles are such that:

4. *Triangle $O_aO_bO_c$ is homothetic to triangle ABC under the homothety $(N, 1/3)$, and the two triangles $G_aG_bG_c$, $O_aO_bO_c$ are symmetric with respect to N.*

As a consequence of this last theorem we have that the circumcenter O_g of triangles $G_aG_bG_c$ is the symmetric of G with respect to N, and is thus the midpoint of GH. This gives us the two following results:

5. *The circumcenter of triangle $G_aG_bG_c$ is the center of the orthocentroidal circle of triangle ABC and is also the orthocenter of triangle $O_aO_bO_c$.*

6. *The centroid of triangle ABC is the circumcenter of triangle $O_aO_bO_c$.*

The circumcircle (O) and the orthocentroidal circle (GH) of triangle ABC have the orthic axis of the triangle for radical axis [2]. Also, the circle (HBC) and the orthocentroidal circle (G_aA) have the orthic axis of triangle HBC as radical axis. But the orthic axes of ABC and HBC pass through the intersection A' of BC and B_hC_h. Now the circumcircles of ABC and HBC are symmetric with respect to BC, and therefore have BC as their radical axis. The point A', then, is the common radical center of the four circles (O), (HBC), (GH), (G_aA). But the radical axis of circles (GH) and (G_aA) is perpendicular to the center line O_gO_a, and thus parallel to BC. Therefore:

7. *The sides BC, CA, AB of triangle ABC are the radical axes of the orthocentroidal circle of ABC and the orthocentroidal circles of triangles HBC, HCA, HAB.*

The radical axis of the orthocentroidal circles (G_bB), (G_cC) is perpendicular to O_bO_c, and therefore parallel to the altitude AH. Now, by the preceding

theorem, vertex A is the radical center of circles (GH), (G_bB), (G_cC). Therefore AH is the radical axis of the circles (G_bB) and (G_cC). Hence:

8. *The altitudes of triangle ABC are the radical axes of the orthocentroidal circles, taken in pairs, of triangles HBC, HCA, HAB.*

As a final consequence we then have:

9. *The orthocenter H of triangle ABC is the radical center of the orthocentroidal circles of triangles HBC, HCA, HAB.*

In closing we might add that some of the preceding results, which we had communicated to Victor Thébault, have been extended by him to the geometry of the tetrahedron [3].

Notes and References

1. The *orthocentroidal circle* of a triangle is the circle having the join of the orthocenter and centroid of the triangle as diameter. It was so named by Robert Tucker.

2. J. Griffiths, Nouvelles Annales de Mathématiques, 1864, p. 345, and 1865, p. 322. Also this MONTHLY [1947, 538] and [1948, 421].

3. V. Thébault has called the sphere having for diameter the join of the Monge point and the centroid of a tetrahedron, the *orthocentroidal sphere* of the tetrahedron. See Notes mathématiques (42), Mathesis, t. 55, 1945–6, p. 264. Properties of this sphere may be found in V. Thébault, Nouvelles sphères associées au tétraèdre, Comptes Rendus des Séances de l'Académie des Science, t. 224, pp. 1614–1616 and V. Thébault, On the Monge point of the tetrahedron, this MONTHLY [1949, 4–13]. In particular, see sec. 6, part (b), of this last paper. In his paper, Sur le cercle et la sphère de Hagge, Bulletin de la Société Royale des Science de Liége, 1946, pp. 44–52, Thébault has designated as the *Hagge sphere* of a point P, with respect to a tetrahedron $T=ABCD$, the circumsphere of the tetrahedron $A_2B_2C_2D_2$, where A_2, B_2, C_2, D_2 are the transforms of the points A_1, B_1, C_1, D_1 in which the cevians through P cut the circumsphere of T under the homotheties $(G_A,-1/2)$, $(G_B,-1/2)$, $(G_C,-1/2)$, $(G_D,-1/2)$, where G_A, G_B, G_C, G_D are the centroids of the faces BCD, CDA, DAB, ABC of T. When P coincides with the centroid G of T, the Hagge sphere becomes the orthocentroidal sphere of T.

A NOTE ON ORTHOPOLAR TRIANGLES*

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Let ABC be a triangle inscribed in a circle (O) of center O and radius R. If the perpendicular to BC drawn through an arbitrary point A' on the circle (O) cuts the circle again in P and, if one draws an arbitrary chord $B'C'$ perpendicular to AP, then the Simson line of A', with respect to triangle ABC, is parallel to AP and triangles ABC, $A'B'C'$ are orthopolar [1]. The Simson lines of A, B, C for triangle $A'B'C'$ and A', B', C' for triangle ABC are concurrent at the midpoint Ω of the segment HH' joining the orthocenters of the two triangles, and Ω coincides with the orthopole of any side of one triangle with respect to the other [2].

* Translated from the French by Howard Eves.

THEOREM I. *The two bisectors of an angle of either of the triangles ABC, $A'B'C'$ determine on any bisector of the other triangle a segment whose midpoint is on circle (O).*

Because triangles ABC and $A'B'C'$ play a symmetric role and because, keeping the vertex letters of one triangle fixed, we may permute the vertex letters of the other, it suffices to show that the two bisectors of angle A determine on the bisectors of angle A' segments whose midpoints are on circle (O). To this end let Q be the point on (O) diametrically opposite to P. Then AQ and $A'Q$ are respectively parallel to $B'C'$ and BC, and the bisectors of angles A and A' of triangle $AA'Q$ are bisectors of the same angles in the triangles ABC and $A'B'C'$. But the two bisectors of any angle of a triangle determine on any other bisector of the triangle a segment whose midpoint is on the circumcircle of the triangle. This proves our theorem.

THEOREM II. *The centers I, I_a, I_b, I_c and I', I_a', I_b', I_c' of the tritangent circles of triangles ABC and $A'B'C'$ are eight points on a common rectangular hyperbola (H).*

In the pencil of rectangular hyperbolas passing through I, I_a, I_b, I_c, the conics consisting of the pairs of bisectors of the angles A, B, C of ABC determine, by Theorem I, on the bisector $A'A_1'$ of angle A' of $A'B'C'$, (A_1' on circle (O)), a symmetric involution with double point A_1'. The points I' and I_a' correspond in this involution. The rectangular hyperbola (H) which passes through the five points I, I_a, I_b, I_c, I' then passes through I_a', and similarly through I_b', I_c'.

THEOREM III. *If we designate by ϕ and ϕ' the foci of the two conics respectively inscribed in triangles ABC and $A'B'C'$ and which have an arbitrary common focus F, the line which joins the foci ϕ_1 and ϕ_1' of the conics inscribed in ABC and $A'B'C'$ and which have a common focus M, arbitrarily situated on $\phi\phi'$, passes through the point F.*

The point ϕ and ϕ' are conjugate points of F with respect to (H) [3]. On the other hand, the foci ϕ_1 and ϕ_1' are on the polar of M with respect to (H). The points ϕ_1 and ϕ_1' are then collinear with F.

THEOREM IV. *If (C_1) and (C_1') are the conics inscribed in triangles ABC and $A'B'C'$ and having Ω for a common focus, then* (1) *the conics have equal non-focal axes,* (2) *the other foci Ω_1 and Ω_1' of the conics are symmetric with respect to the circumcenter O* [4], (3) *the distances of the centers K_1 and K_1' of the conics from the centers N and N' of the nine point circles of the two triangles are equal.*

If d_a, d_b, d_c and d_a', d_b', d_c' are the distances of the orthopole Ω from BC, CA, AB and $B'C'$, $C'A'$, $A'B'$, then [5]

$$d_a d_b d_c = d_a' d_b' d_c'$$

and, consequently [6],

$$\beta^2 = 2Rd_ad_bd_c/\omega = 2Rd_a'd_b'd_c'/\omega = \beta'^2,$$

where β and β' are the non-focal semi-axes of (C_1) and (C_1') and ω is the power of Ω with respect to (O). On the other hand, if ω_1 and ω_1' are the powers of Ω_1 and Ω_1' for circle (O), then [7]

$$\omega\omega_1 = 4\beta^2R^2 = 4\beta'^2R^2 = \omega\omega_1',$$

whence $\omega_1=\omega_1'$ and consequently $O\Omega_1=O\Omega_1'$. That Ω_1, O, Ω_1' are collinear follows from Theorem III by taking $\phi\equiv H$, $\phi'\equiv H'$, $F\equiv O$ and $\phi_1\equiv\Omega_1$, $\phi_1'\equiv\Omega_1'$, $M\equiv\Omega$. The fact that $K_1N=K_1'N'$ now follows by elementary geometry, for K_1K_1' and NN' bisect each other.

THEOREM V. *In a triangle ABC with orthocenter H and circumcenter O, the isogonal conjugates F_1, F_2 of the center of the Taylor circle* [8] *in the medial and orthic triangles are symmetric with respect to the nine point center, and HF_2 is equal and parallel to F_1O.*

For the medial and orthic triangles are orthopolar triangles inscribed in the nine point circle, and the center F of the Taylor circle of ABC is the midpoint of the segment joining the orthocenters of the medial and orthic triangles [9].

Part (2) of Theorem IV is a particular case of the following theorem, due to Blanchard (Le Havre), which is scheduled to appear in *Mathesis*. Guillotin (Le Mans) has given a synthetic demonstration which served as the stimulus for the present note.

THEOREM VI. *Being given two orthopolar triangles ABC and $A'B'C'$ inscribed in a circle, if one designates by ϕ and ϕ' the foci of the two conics respectively inscribed in these triangles and which have a common arbitrary focus F, the line joining the foci ϕ_1 and ϕ_1' of conics respectively inscribed in ABC and $A'B'C'$ and which have for common focus the point dividing the segment $\phi\phi'$ in the ratio k, passes through F and, moreover, the point F divides the segment $\phi_1\phi_1'$ in the ratio k.*

References

1. V. Thébault, Mathesis, 1932, p. 453.
2. T. Lalescu, Gazeta Matematica (Bucarest), 1915, p. 213.
3. Rouché and de Comberouse, Traité de Géométrie, t. II, p. 613.
4. O. Ramler, this MONTHLY, 1940, pp. 140–145 (using complex coordinates).
5. V. Thébault, Mathesis, 1933 (Supplément) and this MONTHLY, 1946, p. 398, problem 4060.
6. J. Neuberg, Nouvelles Annales de Mathématiques, 1870, p. 65.
7. Laguerre, Nouvelles Annales de Mathématiques, 1879, p. 241.
8. *Cf.* R. A. Johnson, Modern Geometry, p. 277.
9. V. Thébault, Mathesis, 1933, *loc. cit.*

MAXIMA AND MINIMA OF AREAS

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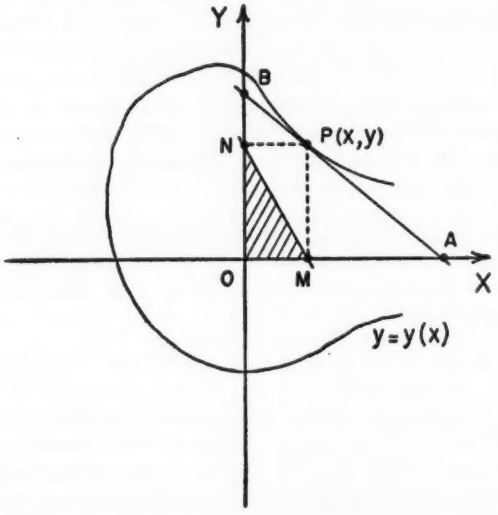
In the accompanying figure we have given the function $y=y(x)$, a point P: (x, y) on the curve, the tangent line to the curve at P cutting the X and Y

axes at A and B respectively, and $OM=x$ and $ON=y$. Let $A_1(x)$ and $A_2(x)$ represent the areas of triangles OAB and OMN respectively. We now define: *a*) a "synonymous" point P at $x=x_1$ denotes a point such that $A_2(x_1)$ and $A_1(x_1)$



are either both at their maximum* values or both at their minimum values; *b*) an "antonymous" point P at $x=x_1$ denotes a point such that either $A_2(x_1)$ is maximum and $A_1(x_1)$ is a minimum or $A_2(x_1)$ is minimum and $A_1(x_1)$ is maximum. We shall prove the following theorems which hold for P anywhere in the plane except on the axes; the latter cases lead to difficulties and trivial cases requiring special studies.

THEOREM 1. *If the solution $x=x_1$ of $A_2'(x)=0$ is such that $A_2''(x_1)\neq 0$ and $y'''(x_1)$ exists, then*

(a) $y(x_1)y''(x_1)>0$ *implies that P at $x=x_1$ is a synonymous point;*

(*b*) $y(x_1)y''(x_1)<0$ *implies that P at $x=x_1$ is an antonymous point;*

(*c*) $y''(x_1)=0$ *implies that $A_1(x_1)$ is neither maximum nor minimum, provided* $y'''(x_1)\neq 0$.

THEOREM 2. *If the solution $x=x_1$ of $A_2'(x_1)=0$ is such that $A_2''(x_1)=\cdots=A_2^{(2h-1)}(x_1)=0$, $A_2^{(2h)}(x_1)\neq 0$, $y^{(2h)}(x_1)$ and $y^{(2h+1)}(x_1)$ exist $(h>1)$, then P at $x=x_1$ is a synonymous point.*

Proofs: By means of intercepts of AB, we find that

$$2A_1 = -(xy'-y)^2/y'. \tag{1}$$

* Maxima and minima values for $f(x)$ are here used in the sense that at $x=x_1$, $f'(x_1)=\cdots=f_2^{(2h-1)}(x_1)=0$ and $f^{(2h)}(x_1)\neq 0$.

Obviously,

$$2A_2 = xy, \tag{2}$$

$$2A_2' = xy' + y. \tag{3}$$

Eliminating y and y' from (1), (2), and (3), we find

$$A_1 = 4A_2 + x^2A_2'^2/(A_2 - xA_2'). \tag{4}$$

Differentiating, we find that $A_1'(x_1)=0$ when $A_2'(x_1)=0$ provided $A_2''(x_1)$ exists, which is easily justified by the existence of $y''(x_1)$ and $A_2''=xy''+2y'$. Differentiating again, we find

$$A_1''(x) = 4A_2''(x) + \sum \frac{2}{e_1!e_2!} D^{e_1}(x^2A_2'^2)D^{e_2}(A_2 - xA_2')^{-1}, \quad (e_1 + e_2 = 2), \tag{5}$$

or

$$A_1''(x_1) = 4A_2''(x_1) + 2x_1^2\{A_2'(x_1)\}^2/A_2(x_1), \tag{6}$$

in which $y''(x_1)$ and $y'''(x_1)$ exist by hypothesis

Eliminating x_1 and $A_2(x_1)$ by means of (3) and (2) and replacing one $A_2''(x_1)$ on the right by $\frac{1}{2}(x_1y''(x_1)+2y'(x_1))$, in (6), we obtain

$$A_1'' = 2A_2''yy''/y'^2 \text{ at } x=x_1 \tag{6'}$$

or

$$A_1''A_2'' = 2(A_2''/y')^2yy'' \quad \text{at} \quad x = x_1. \tag{7}$$

Therefore

$$\text{sign } (A_1''A_2'') = \text{sign } (yy'') \quad \text{at} \quad x = x_1.$$

This proves parts (a) and (b) of Theorem 1. In case $y''(x_1)=0$, we differentiate (1) directly and find that

$$2A_1''' = 4y'''(y - xy')/y' \quad \text{at} \quad x = x_1. \tag{8}$$

Now if $y(x_1)-x_1y'(x_1)=0$, we have, with $A_2'(x_1)=0$, the contradiction that $y(x_1)=0$. Hence $A_1'''(x_1)\neq 0$ unless $y'''(x_1)=0$, which proves part (c) of Theorem 1.

We next turn to the case $A_2''(x_1)=\cdots=A_2^{(2h-1)}(x_1)=0$ and $A_2^{(2h)}(x_1)\neq 0$. This also gives us a critical value for $A_2(x_1)$. We employ

$$D^nA_1(x_1) = 4D^nA_2(x_1) + \sum_{i=0}^{n} \binom{n}{i} D^i(x^2A_2'^2)D^{n-i}(A_2 - xA_2')^{-1}, (n=3, \cdots, 2h). \tag{9}$$

By induction, we find that $D^{n-i}(A_2-xA_2')^{-1}$ is a polynomial expression divided by a power of (A_2-xA_2'). Also, $D^i(x^2A_2'^2)$ is zero at $x=x_1$. Hence

$$A_1^{(n)}(x_1) = 4A_2^{(n)}(x_1), \tag{10}$$

provided $A_2^{(2h)}(x_1)$ and $A_2^{(2h+1)}(x_1)$ exist. But repeated differentiations of (3) equated to zero and the use of the existence of $y^{(2h)}(x_1)$ and $y^{(2h+1)}(x_1)$ justify this provision. Hence (10) implies Theorem 2.

It is interesting to note that, at these critical values, the generating lines AB and MN are parallel.

A SPECIAL TETRAHEDRON

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The following sections represent a continuation of the paper under the same title that appeared in this MONTHLY, volume 56, May, 1949, page 312.

7. The place of the Monge point on the edge of a tetrahedron $(\sigma\sigma)$. The Monge point of any tetrahedron $(T) = DABC$ is the center of the equilateral hyperboloid (H) determined by the altitudes of (T), and (T) is inscribed in (H).* Now in the case of a doubly special tetrahedron $(\sigma\sigma)$ (§4) the Monge point M lies on an edge, say BC, of (T); the points B, C of (H) are therefore symmetrical with respect to the center M of (H); that is, M coincides with the mid-point A' of the edge BC. Thus: *If the Monge point of a tetrahedron* (T) *lies on an edge of* (T), *then the Monge point coincides with the mid-point of that edge.*

8. The circumcenter of a doubly special tetrahedron $(\sigma\sigma)$. (a) The centroid G of any tetrahedron bisects the segment MO determined by the Monge point M and the circumcenter O; and G also bisects the segment determined by the mid-points of any pair of opposite edges. In the case of a tetrahedron $(\sigma\sigma)$ the point M coincides with the mid-point A' of the edge BC (§7); therefore the point O coincides with the mid-point A'' of the opposite edge DA. Thus: *If an edge of a tetrahedron* (T) *passes through the Monge point of* (T), *the opposite edge passes through the circumcenter of* (T).

(b) *Conversely, if an edge of a tetrahedron* (T) *passes through the circumcenter of* (T), *then the mid-point of the opposite edges is the Monge point of* (T).

The proof is the same as of the direct proposition, the rôles of the points M, O being interchanged.

(c) Thus to construct a tetrahedron $(\sigma\sigma)$ it is necessary and sufficient to take for two of the vertices D, A two diametrically opposite points on a sphere (O), and for the other two vertices B, C any two other points of (O), and the line BC does not have to be perpendicular to DA, as stated in §5. It was erroneous to assume that the Simson lines of the points D_h, A_h for the triangles ABC, DBC, respectively, and the line BC, have a determinate point in common; these three lines coincide.

* H. Schroeter, Flaechen zweiter Ordnung, pp. 203 ff., 1880, George Salmon, Analytic geometry of three dimensions, vol. I, p. 140, fifth edition, 1912.

(*d*) The propositions (*a*) and (*b*) above may be considered as properties of an equilateral hyperboloid, and as such may be considered as generalizations of the properties of four concyclic points of an equilateral hyperbola.*

NOTE ON THE LAURENT SERIES

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1. Introduction. Let $f(x)$ be a uniform analytic function of x, and let it be represented in the region $|x|<1$ by the convergent power series,

$$f(x)=\sum_{n=0}^{\infty} a_n x^n. \tag{1}$$

Suppose $x=1$ is an isolated singular point of f. There is then a Laurent series for $f(x)$:

$$f(x)=\sum_{l=0}^{\infty} \alpha_l(1-x)^{-l}+\sum_{l=1}^{\infty} \beta_l(1-x)^l. \tag{2}$$

The Taylor coefficients $\{a_n\}$ completely determine the function f, and therefore determine the Laurent coefficients $\{\alpha_l\}$, $\{\beta_l\}$. If the Taylor coefficients are given, how can one actually calculate the Laurent coefficients? Solution of this problem would enable one to compute the values of $f(x)$ outside of its original region of definition by utilizing the Laurent series.

This problem will be solved in the simplest case in which $x=1$ is the *only* singularity of $f(x)$. Since f is thereby assumed to be regular at infinity, the $\{\beta_l\}$ necessarily vanish, and we have

$$f(x)=\sum_{0}^{\infty} \alpha_l(1-x)^{-l}, \qquad x \neq 1. \tag{3}$$

2. Calculation. The obvious procedure is to expand $(1-x)^{-l}$ in powers of x, and rearrange (3) as a power series in x, which can then be compared coefficientwise with (1), since both expansions represent $f(x)$ in $|x|<1$. Thus,

$$\begin{aligned}\sum_{0}^{\infty} \alpha_l(1-x)^{-l} &= \alpha_0+\sum_{l=1}^{\infty} \alpha_l \sum_{m=0}^{\infty}\binom{l+m-1}{m} x^m \\ &= \alpha_0+\sum_{m=0}^{\infty} x^m \sum_{l=1}^{\infty}\binom{l+m-1}{m}\alpha_l, \quad |x|<1,\end{aligned}$$

on the assumption that the interchange of summation orders is legitimate. Then, by comparison with (1),

$$a_0=\alpha_0+\sum_{1}^{\infty} \alpha_l,$$

* John Casey, Analytic geometry, p. 291, Nos. 6 and 7, sec ed., 1893; Mathesis, 1928, p. 136.

(4)
$$a_n = \sum_{l=1}^{\infty} \binom{l+n-1}{n} \alpha_l, \qquad n \geqq 1.$$

In order to obtain $\{\alpha_l\}$, we should have to solve this system of linear equations in infinitely many variables.

However, this difficulty can easily be avoided. The function $f(x)$ is an entire function of $(1-x)^{-1}$ but may be considered an entire function of $x(1-x)^{-1} = (1-x)^{-1}-1$. Therefore, we have an expansion, valid for $x \neq 1$,

(5)
$$f(x) = e_0 + \sum_{l=1}^{\infty} e_l x^l (1-x)^{-l};$$

and for $|x| < 1$, we expand the denominators,

(6)
$$f(x) = e_0 + \sum_{l=1}^{\infty} e_l x^l \sum_{m=0}^{\infty} \binom{l+m-1}{m} x^m = e_0 + \sum_{n=1}^{\infty} x^n \sum_{l=1}^{n} e_l \binom{n-1}{l-1}.$$

Since the sum in (5) is a power series in $u = x(1-x)^{-1}$ which converges for all finite values of u, we have $\lim |e_l|^{1/l} = 0$, or $|e_l| < \epsilon^l$ for $l > l_0(\epsilon)$. With this estimate it is easy to justify the inversion of summation orders in (6).

Comparing the last series in (6) with (1), we find

(7)
$$a_0 = e_0,$$
$$a_n = \sum_{l=1}^{n} e_l \binom{n-1}{l-1}, \qquad n \geqq 1.$$

This is a linear system which obviously determines $\{e_n\}$ uniquely, since the equations form a triangular set.

Now we wish to find the relation between the $\{e_n\}$ and the $\{\alpha_n\}$. In the series in (5) write

$$x^l = (1 + x - 1)^l = \sum_{j=0}^{l} \binom{l}{j} (x-1)^j;$$

then

$$\begin{aligned} f(x) &= e_0 + \sum_{m=1}^{\infty} e_m \sum_{j=0}^{m} \binom{m}{j} (-1)^j (1-x)^{j-m} \\ &= e_0 + \sum_{m=1}^{\infty} e_m (1-x)^{-m} + \sum_{j=1}^{\infty} (-1)^j \sum_{m=j}^{\infty} \binom{m}{m-j} (1-x)^{j-m} e_m \\ &= e_0 + \sum_{l=1}^{\infty} e_l (1-x)^{-l} + \sum_{l=0}^{\infty} (1-x)^{-l} \sum_{m=l+1}^{\infty} (-1)^{m+l} \binom{m}{l} e_m \\ &= e_0 + \sum_{m=1}^{\infty} (-1)^m e_m + \sum_{l=1}^{\infty} (1-x)^{-l} \sum_{m=l}^{\infty} (-1)^{m+l} \binom{m}{l} e_m. \end{aligned}$$

The last series is in the form (3); hence

$$\alpha_0 = e_0 + \sum_{m=1}^{\infty} (-1)^m e_m,$$

$$\alpha_l = \sum_{m=l}^{\infty} (-1)^{m+l} \binom{m}{l} e_m, \qquad l \geqq 1. \tag{9}$$

The last step is to express the $\{e_n\}$ in terms of the $\{a_n\}$ by means of (7). This is easily done if we make use of the "orthogonality" property

$$\sum_{n=l}^{m} (-1)^{n+m} \binom{n-1}{l-1} \binom{m-1}{n-1} = \delta_{lm}. \tag{10}$$

In order to prove (10) we note that

$$\binom{n-1}{l-1}\binom{m-1}{n-1} = \binom{m-1}{l-1}\binom{m-l}{m-n},$$

so that the sum in (10) equals

$$\binom{m-1}{l-1} \sum_{n=l}^{m} (-1)^{n+m} \binom{m-l}{n-l}.$$

In the above sum we set $p=n-l \geqq 0$; p then runs from 0 to $m-l$. Hence,

$$\sum_{n=l}^{m} (-1)^{n+m} \binom{n-1}{l-1}\binom{m-1}{n-1} = (-1)^{l+m} \binom{m-1}{l-1} \sum_{p=0}^{m-l} (-1)^p \binom{m-l}{p}.$$

The sum in the right member of the last equation vanishes unless $m-l=0$, in which case the right member becomes equal to 1. This proves (10).

Now multiply both members of the second equation of (7) by $(-1)^{n+m}\binom{m-1}{n-1}$ and sum on n, $n=1, 2, \cdots, m$. Interchanging the order of summation and using (10) we get

$$e_m = \sum_{n=1}^{m} (-1)^{n+m} \binom{m-1}{n-1} a_n, \qquad m \geqq 1, \tag{11}$$

and we already had $e_0=a_0$ from the first equation of (7). Using these results in (9) we obtain the final formula:

$$\alpha_0 = a_0 + \sum_{m=1}^{\infty} \sum_{n=1}^{m} (-1)^n \binom{m-1}{n-1} a_n,$$

$$\alpha_l = \sum_{m=l}^{\infty} \binom{m}{l} \sum_{j=1}^{m} (-1)^{l+j} \binom{m-1}{j-1} a_j, \qquad l \geqq 1. \tag{12}$$

The author wishes to thank the referee, who showed that α_0 could be expressed in the above form.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College and Massachusetts Institute of Technology

All material for this department should be sent to C. B. Allendoerfer, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

PARTIAL FRACTIONS WITH REPEATED LINEAR OR QUADRATIC FACTORS

M. R. SPIEGEL, Rensselaer Polytechnic Institute

The great tedium involved in separating into partial fractions a rational algebraic function in which the denominator has repeated linear or quadratic factors has long been recognized. It is the purpose of this article to indicate a method which the author has found convenient in working with Laplace transforms where such separation into partial fractions is important from the practical viewpoint.

The best way to present the ideas involved in this labor saving device is by means of an example. We pose the problem:

Separate

$$\frac{3x+1}{(x-2)(x^2+x+1)^3}$$

into partial fractions.

Normally one would write

$$\frac{3x+1}{(x-2)(x^2+x+1)^3} \equiv \frac{A}{x-2} + \frac{Bx+C}{(x^2+x+1)^3} + \frac{Dx+E}{(x^2+x+1)^2} + \frac{Fx+G}{x^2+x+1}.$$

Since A can be calculated immediately (by the usual method of multiplying through in this identity by $(x-2)$ and then letting $x \to 2$) we have 6 coefficients to compute, *i.e.*, we have to solve 6 equations in 6 unknowns. Anyone who has used this method must realize how much labor is involved.

Suppose, however, we consider

$$\frac{3x+1}{(x-2)(x^2+x+a)} \equiv \frac{A}{x-2} + \frac{Bx+C}{x^2+x+a} \tag{1}$$

where a is a parameter and A, B, and C are of course functions of a.

Then

$$A = \frac{7}{a+6}.$$

Multiplying (1) through by x and letting $x\to\infty$ we observe that

$$0 = A + B \quad \text{or} \quad B = \frac{-7}{a+6}.$$

Finally letting $x=0$ in (1) we find

$$-\frac{1}{2a} = -\frac{1}{2}\left(\frac{7}{a+6}\right) + \frac{C}{a}$$

or

$$C = -\frac{1}{2} + \frac{1}{2}\left(\frac{7a}{a+6}\right) = -\frac{1}{2} + \frac{1}{2}\left(\frac{7a+42-42}{a+6}\right)$$
$$= -\frac{1}{2} + \frac{7}{2} - \frac{21}{a+6} = 3 - \frac{21}{a+6}.$$

We now practically have the problem solved.
Differentiating (1) twice with respect to a we get

$$\frac{2(3x+1)}{(x-2)(x^2+x+a)^3} \equiv \frac{A''}{x-2} + \frac{B''x+C''}{x^2+x+a} - \frac{2(B'x+C')}{(x^2+x+a)^2} + \frac{2(Bx+C)}{(x^2+x+a)^3} \tag{2}$$

where the primes denote differentiations with respect to a. In view of the fact that

$$A' = \frac{-7}{(a+6)^2} \qquad B' = \frac{7}{(a+6)^2} \qquad C' = \frac{21}{(a+6)^2},$$
$$A'' = \frac{14}{(a+6)^3} \qquad B'' = \frac{-14}{(a+6)^3} \qquad C'' = \frac{-42}{(a+6)^3}.$$

Putting $a=1$ and using the values thus obtained in (2) we have the result desired.

Further extensions of this and applications to linear repeated factors are obvious.

MORE STRESS ON GENERAL FORMULATIONS IN CALCULUS PROBLEMS

LUISE LANGE, Woodrow Wilson Junior College, Chicago

In many calculus texts problems are formulated too one-sidedly in terms of particular, numerical data rather than in general terms. While pedagogically it may be wise to begin a new type of problem with some numerical examples, it is only the general formulation, and the interpretation of the answer in general

terms, which can give insight into the functional relation between the given and the derived data.

We illustrate our point by some examples from the traditional "minimum fence" and "related time rate" problems. They could readily be multiplied.

1. *Numerical Formulation.* A rectangular lot of given area is to be fenced in on four sides. For what dimensions of the lot will the length of the fence be a minimum? Answer: a square. Same question, but the lot is to be fenced in on only three sides. Answer: (with one side x and two sides y) $x:y=2:1$. Same question, but the lot is to be divided internally by three fences. Answer: (with inside fences parallel to x) $x:y=2:5$. Same question, but the material of one fence costs twice as much as that of the three others. Answer: (with higher-cost material in x direction) $x:y=3:2$.

General formulation. The lot is to be fenced in by n fences (inside or outside) in one direction, and m in the other. The minimum fence is that for which $x:y=m:n$, or $nx=my$; which means, if just as much fence runs in the one direction as in the other. If instead of length it is cost which is to be minimized the condition is the same: the cost is minimum if all fences in one direction cost as much as those in the other.—In fact, all these several questions are one: the minimum of the function $F=c_1x+(c_2A/x)$, which lies at $c_1-c_2(A/x^2)=0$ or $c_1x=c_2(A/x)$. The graphical presentation of this function throws additional light on the question. F is the sum of the ordinates of the straight line $y=c_1x$, which represent the length or cost of the fences in one direction, and of the hyperbola $y=c_2A/x$ representing the length or cost of the fences in the other direction. The minimum of the function F lies at the intersection of the straight line with the hyperbola.

Posed in general formulation the problem of the minimum surface of a closed or open cylindrical can of given volume receives a similarly comprehensive answer. If there are n bases (e.g. inside partitions as well as bottom and top) the minimum condition is found to be $h=nr$, which, substituted into the formula for the lateral surface is seen to mean that the minimum surface is that for which the lateral surface is twice as great as that of all bases.

2. *Numerical formulation.* A street light hangs 18 ft. above the the street. A man 6 ft. tall walks along the street at a rate 3 ft/sec. At what rate is his shadow lengthening? Answer 1.5 ft/sec.

General formulation. The light hangs H ft. high. An object h ft high moves along the street at rate v ft/sec. At what rate is its shadow lengthening? Answer:

$$ds/dt = \frac{h}{H-h}\,v = \frac{1}{H/h-1}\,v.$$

Interpretation: (a) The rate of the shadow's lengthening is proportional to the rate at which the object moves. (b) The factor of proportionality depends only on the ratio H/h. If h gets small in comparison to H the factor approaches zero (e.g. a mouse, or even more so an ant, running along the street push their practically constant shadows ahead of them). As H/h approaches 1 the shadow

lengthens ever more rapidly (e.g. a truck 17½ ft. high under a light 18 ft. high has its shadow lengthening 35 times as fast it drives). (c) The relative rate of the shadow's lengthening, on the other hand,

$$\frac{\frac{ds}{dt}}{s} = \frac{v}{x},$$

does not depend on H and h, but is, for a given v, inversely proportional to the distance x from the light. (A mouse and an elephant walking side by side at the rate 2.5 ft/sec. have their respective shadows lengthening at the same rate 10 percent per second when 25 ft. away from the light, and only 1 percent per second when 250 ft. away from the light.)

We still wish to add that positively harmful, in our opinion, are the numerical specifications not only for arbitrary constants but for variables of which the required datum is a function, merely for the purpose of "coming out" with a numerical answer. Example: from a bridge 30 ft. high a boat is towed in by a rope which is pulled in at a rate 4 ft/sec. At what rate is the boat moving *when 40 ft. away from the bridge*? The last specification is pedagogically very bad, because it confuses in the student's mind the basic question as to what is constant and what variable in the problem.

EQUIVALENT PROPERTIES OF A RING

R. A. Beaumont, University of Washington

The well-known theorem* which states that if R is a commutative ring which contains no proper divisors of zero, then every polynomial with coefficients in R has at most n zeros in R, is usually accompanied by examples to show that neither hypothesis can be omitted. The fact that the converse holds, although the proof is almost trivial, is not given. It may be of some interest to record the following equivalent properties of a ring.

Theorem. *In any ring R, the following properties are equivalent.*

(1) *R is commutative and contains no proper divisors of zero.*

(2) *Every polynomial of degree n with coefficients in R has at most n zeros in R.*

(3) *Every linear polynomial with coefficients in R has at most one zero in R.*

(4) *If a, b, c are in R, $b \neq 0$, then $ba = cb$ implies $a = c$.*

Proof. (1) implies (2) is the theorem mentioned above and (2) implies (3). Let a, b, c be elements of R such that $b \neq 0$ and $ba = cb$. Then the linear equation $bx - ba = 0$ has a as a right zero and c as a left zero and $a = c$ by (3). Thus (3) implies (4). If (4) is satisfied, assume that $ba = 0$ where $b \neq 0$. Then $ba = 0 = 0b$ implies $a = 0$ so that R has no proper divisors of zero. Further $b(ab) = (ba)b$ implies $ab = ba$ for all a in R and $b \neq 0$ in R. Since the latter equation is satisfied for $b = 0$, R is commutative. Hence (4) implies (1).

* See for example, B. L. van der Waerden, Modern Algebra, Vol. 1, Ungar (1949), page 65.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 906. *Proposed by Vern Hoggatt, Oregon State College*

Let S be a bounded infinite point set. Show that there exists a polygonal path of arbitrarily small length passing through an infinite number of points of S.

E 907. *Proposed by C. D. Olds, San Jose State College*

A derangement of n ordered letters is a permutation of the n letters which leaves no letter fixed. Find formulas which will give the number of odd and the number of even derangements of n ordered letters.

E 908. *Proposed by R. J. Koch, Tulane University*

Show that $r=1/2$ is the only rational value of r for which the geometric series Σar^n, $a\neq 0$, has the property that there exists a term equal to the sum of all the terms from some point on, and that when $r=1/2$ this property holds for every term.

E 909. *Proposed by W. E. Buker, Perry High School, Pittsburgh, Pa.*

Find positive integers satisfying

$$N = p^2 + k_1^2 = (p+1)^2 + k_2^2 = (p+2)^2 + k_3^2.$$

E 910. *Proposed by R. M. Gordon, Carnegie Institute of Technology*

If $k>0$, $f_i(x)$ are independent, $f_i^{(n)}(x)$ exist and are well behaved, find

$$\lim_{h\to 0} h^{-k} \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1(x+h) & f_2(x+h) & \cdots & f_n(x+h) \\ f_1(x+2h) & f_2(x+2h) & \cdots & f_n(x+2h) \\ \cdot & \cdot & \cdots & \cdot \\ f_1[x+(n-1)h] & f_2[x+(n-1)h] & \cdots & f_n[x+(n-1)h] \end{vmatrix}.$$

SOLUTIONS

A Three Man Dice Game

E 850 [1949, 31]. *Proposed by L. C. Hsu, National Tsing-Hua University, Peiping, China*

Three persons, A, B, and C, in rotation throw a pair of dice. If the points 6, 7, 8 are consecutively gotten by A, B, C, then A is declared winner; if the points 7, 8, 6 are consecutively gotten by B, C, A, then B is declared winner; if the points 8, 6, 7 are consecutively gotten by C, A, B, then C is declared winner. Find their chances of winning.

Solution by the Proposer. Let us call the three consecutive throws numbered $3n-2$, $3n-1$, $3n$ the nth cycle with respect to A, those numbered $3n-1$, $3n$, $3n+1$ the nth cycle with respect to B, and those numbered $3n$, $3n+1$, $3n+2$ the nth cycle with respect to C. Define $P_A(n)$, $P_B(n)$, $P_C(n)$ as the probabilities that A, B, C win exactly in their respective nth cycles. Denote by P_A, P_B, P_C the total probabilities of A, B, C winning. It is clear that for $m \neq n$, $P_A(m)$ and $P_A(n)$ are disjoint. Moreover, the events whose probabilities are given by $P_A(n)$, $P_B(m)$, $P_C(k)$ are also incompatible. Therefore

$$P_A = \sum_{n=1}^{\infty} P_A(n), \qquad P_B = \sum_{n=1}^{\infty} P_B(n), \qquad P_C = \sum_{n=1}^{\infty} P_C(n),$$

where

$$P_A(1) = (5/36)(6/36)(5/36),$$
$$P_B(1) = (31/36)(6/36)(5/36)^2,$$
$$P_C(1) = (30/36)(5/36)^2(6/36).$$

Now it is easy to obtain the following equations, for $n \geqq 2$:

$$(1)\ P_A(n) = \left\{1 - \sum_{k=1}^{n-2} P_A(k) - \sum_{k=1}^{n-2} P_B(k) - \sum_{k=1}^{n-2} P_C(k)\right\} (31/36)(5/36)(6/36)(5/36),$$

$$(2)\ P_B(n) = \left\{1 - \sum_{k=1}^{n-1} P_A(k) - \sum_{k=1}^{n-2} P_B(k) - \sum_{k=1}^{n-2} P_C(k)\right\} (31/36)(6/36)(5/36)(5/36),$$

$$(3)\ P_C(n) = \left\{1 - \sum_{k=1}^{n-1} P_A(k) - \sum_{k=1}^{n-1} P_B(k) - \sum_{k=1}^{n-2} P_C(k)\right\} (30/36)(5/36)(5/36)(6/36).$$

From (1) and (2) we then have

$$\sum_{n=2}^{\infty} P_A(n) - \sum_{n=2}^{\infty} P_B(n) = (4650/36^4) \sum_{n=2}^{\infty} P_A(n-1),$$

and from (2) and (3)

$$30 \sum_{n=2}^{\infty} P_B(n) - 31 \sum_{n=2}^{\infty} P_C(n) = (139500/36^4) \sum_{n=2}^{\infty} P_C(n-1).$$

Using the fact that $P_A + P_B + P_C = 1$, we finally get, by solving approximately,

$$P_A \doteq 0.33818, \qquad P_B \doteq 0.33680, \qquad P_C \doteq 0.32502.$$

Other solutions were given by Louis Berkofsky, R. E. Edwards, H. E. H. Greenleaf, J. M. Kingston, Sam Kravitz, Roger Lessard, Bart Park, and L. A. Ringenberg.

Infinite Radicals

E 874 [1949, 404]. *Proposed by C. S. Ogilvy, Trinity College*

If $y_1=x^{1/2}$, $y_2=(x+y_1)^{1/2}$, $\cdots$, $y_m=(x+y_{m-1})^{1/2}$, prove that $\lim_{m\to\infty}y_m$ is integral if and only if x is of the form $n(n-1)$, $n=2, 3, \cdots$, in which case the limit is n.

Solution by C. W. Trigg, Los Angeles City College. The given sequence is monotone increasing and convergent for $x>0$. (See, for example, Aaron Herschfeld, *On Infinite Radicals*, this MONTHLY, pp. 419–29, 1935.) In general, the sequence $y_1=x^{1/p}$, $y_2=(x+y_1)^{1/p}$, $\cdots$, $y_m=(x+y_{m-1})^{1/p}$ is convergent for $p\geqq 2$, since it is convergent for $p=2$. Denote $\lim_{m\to\infty}y_m$ by η, whereupon we have $\eta^p-\eta-x=0$, which has exactly one positive root for $x>0$. For integer values of the limit η, $x=\eta^p-\eta$. For $p=2$, $x=\eta(\eta-1)$; and for $p=3$, $x=(\eta-1)\eta(\eta+1)$.

Also solved by W. D. Berg, D. H. Browne, M. S. Klamkin, Martin Milgram, L. A. Ringenberg, Azriel Rosenfeld, and the proposer.

See E 474 [1942, 197].

Property of an Arithmetic Progression

E 876 [1949, 473]. *Proposed by H. L. Lee, University of Tennessee*

The product of four consecutive terms of an arithmetic progression of integers plus the fourth power of the common difference is a perfect square but in no case a perfect fourth power.

Solution by R. V. Andree, University of Oklahoma. Let a be the first term of the four, and d the common difference. Then

$$a(a+d)(a+2d)(a+3d)+d^4=(a^2+3ad+d^2)^2,$$

a perfect square. However, that it is never a perfect fourth power is false. A counter example is obtained with $a=3$, $d=7$. Other counter examples may be constructed by taking

$$a=u^2-v^2, \qquad d=2uv+3v^2,$$

in which case

$$a(a+d)(a+2d)(a+3d)+d^4=(u^2+3uv+v^2)^4.$$

Also solved by Max Beberman, Daniel Block, Leslie Booth, D. H. Browne, Aaron Buchman, W. E. Buker, J. H. Butchart, E. H. Clarke, E. J. Cogan, W. J. Feeney, W. Fulks, Vern Hoggatt, R. T. Hood, P. W. M. John, Harris Jones, M. S. Klamkin, Frank Kocher, Sam Kravitz, Sidney Kravitz, F. T. Leahy, Jr., Emma Lehmer, D. W. Matlack, F. L. Miksa, E. P. Miles, Jr., T. F.

Mulcrone, C. S. Ogilvy, S. T. Parker, J. W. Ponds, P. W. A. Raine, L. B. Rall, K. S. Rao, R. J. Reid, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, R. W. Shoemaker, R. V. Simpson, Guy Stevenson, B. M. Stewart, C. W. Trigg, Robert Ullman, G. W. Walker, Lila Walker, Hazel Wilson, and the proposer.

Eleven of these solutions failed to note that the second part of the problem as quoted is incorrect. The failure was due to assuming that a form which is not *formally* a square is never a square in integers. The solution by G. W. Walker constituted a remarkably complete discussion of the problem.

A. Desboves, *Nouv. Ann. Math.* (2), 18, 1879, 269 and (3) 5, 1886, 226–33, showed that $x^2+dxy+by^2=z^2$ has for complete solution

$$x = \pm (q^2 - bp^2), \qquad y = \pm (dp^2 + 2pq), \qquad z = \pm (q^2 + bp^2 + dpq).$$

This yields, essentially, the general solution given above by Andree. See Dickson, *History of the Theory of Numbers*, v. III, pp. 404–407.

Property of Concurrent Cevians

E 879 [1949, 474]. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Let S_1, S_2, S_3 be the midpoints of three concurrent cevians of triangle ABC. Let S_2S_3, S_3S_1, S_1S_2, meet the sides BC, CA, AB in A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3 respectively. Show that (1) A_2, A_3; B_3, B_1; C_1, C_2 are isotomic points on the segments BC, CA, AB, (2) A_1, B_2, C_3 are collinear, (3) A_2, A_3, B_3, B_1, C_1, C_2 lie on a conic.

Solution by J. H. Butchart, Arizona State College at Flagstaff. Triangles ABC and $S_1S_2S_3$ are perspective. Hence A_1, B_2, C_3 are collinear. The triangle formed by joining the midpoints A', B', C' of the sides of triangle ABC is circumscribed about triangle $S_1S_2S_3$ and is inscribed and perspective with triangle ABC. Hence triangles $A'B'C'$, $S_1S_2S_3$ are perspective. Then the pencil $S_1(S_2S_3, A'B')$ is harmonic and A' is the midpoint of A_2A_3. The conics through the fixed points B_1, B_3, C_2, C_1 establish an involution on the line BC, which, for the degenerate cases, pairs B with C and A_1 with its isotomic conjugate. Hence A_2, A_3 are determined by the same conic.

Also solved by L. M. Kelly, Roscoe Woods, and the proposer.

Editorial Note. The perspectivity of triangles $A'B'C'$ and $S_1S_2S_3$ can be established by Ceva's theorem, for

$$(B'S_1 \cdot C'S_2 \cdot A'S_3)/(S_1C' \cdot S_2A' \cdot S_3B') = (BL \cdot CM \cdot AN)/(LC \cdot MA \cdot NB) = 1,$$

where L, M, N are the feet, on the sides BC, CA, AB, of the given concurrent cevians. Also, part (3) follows from part (1) by Carnot's theorem.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4385. *Proposed by P. Ungar, University College, London, England*

The three-digit sections of the sequence 1110001011 represent all three-digit numbers in the binary system exactly once each. For a given positive integer n an analogous sequence is obtained in the following manner: write down n 1's to begin with, and in each subsequent place write 0 unless the n-digit section thus completed occurs previously, in which case put 1. Show that the resulting sequence of 2^n+n-1 digits has the same property as the case $n=3$ cited at the outset.

4386. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In any triangle the Monge circle (orthoptic locus) of the inscribed conic concentric with the circumcircle is orthogonal to the polar circle.

4387. *Proposed by Paul Bateman, Institute for Advanced Study*

If $\sigma_r(n)$ denotes the sum of the rth powers of the divisors of the positive integer n, prove that

$$\sigma_r(n)\sigma_r(m) = \sum_{d\mid(n,m)} d^r\sigma_r(nm/d^2),$$

where d runs through all the common divisors of n and m.

4388. *Proposed by Paul Erdös, Syracuse University and W. H. J. Fuchs, Cornell University*

Let

$$f(z) = \prod_{i=1}^{n} (z - z_i), \qquad |z_i| \leqq 1.$$

Consider the set $|f(z)| \leqq 1$. Prove that it consists of at most $n-1$ components.

4389. *Proposed by F. J. Dyson, Institute for Advanced Study*

Given N numbers a_m satisfying the N equations

$$\sum_{m=1}^{N} \frac{a_m}{m+n} = \frac{4}{2n+1}, \qquad n = 1, 2, \cdots, N.$$

prove that

$$\sum_{m=1}^{N} \frac{a_m}{2m+1} = 1 - \frac{1}{(2N+1)^2}.$$

This problem arose in an investigation of the diffraction of light (H. Levine and J. Schwinger, *Physical Review*, 74, 970 (1948)).

SOLUTIONS

A Property of the Orthocentric Tetrahedron

4266 [1947, 479]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a tetrahedron $ABCD$ and a sphere (S). If the polar planes of the vertices A, B, C, D with respect to (S) and the corresponding planes tangent to the circumsphere at A, B, C, D cut each other, respectively, on the faces BCD, CDA, DAB, ABC, the tetrahedron is orthocentric. Establish a converse theorem.

*Solution by the Proposer.** In the coaxal pencil of spheres determined by the circumsphere (O) of the tetrahedron $ABCD$ and the given sphere (S), let (Σ) be the sphere which separates A and B harmonically. The polar plane of the vertex A with respect to (Σ) contains the line Δ_a of the plane BCD in which the polar planes of A with respect to (O) and (S) intersect (since the polar planes of a point with respect to the spheres of a coaxal pencil form a coaxal pencil). Likewise the polar plane of A with respect to (Σ) contains the point B, and so it coincides with BCD. Hence (Σ) separates A and C, and A and D, harmonically. The same reasoning holds for the other vertices B, C, D. Therefore (Σ) is the conjugate sphere of the tetrahedron $ABCD$ and the given tetrahedron is orthocentric.

Conversely, in every orthocentric tetrahedron the property of (S) holds for every sphere of the coaxal pencil determined by the circumsphere and the conjugate sphere.

Difference of Consecutive Cubes

4299 [1948, 321]. *Proposed by R. C. Lyness, Crakemarsh, Uttoxeter, Staffordshire, England*

If the difference between two consecutive cubes is a square, then it is the square of the sum of two successive squares.

Solution by Ernest Trost, Technikum Winterthur, Switzerland. We have to solve the Diophantine equation

$$(x+1)^3 - x^3 = 3x^2 + 3x + 1 = y^2. \tag{1}$$

Multiplication by 4 gives

$$3(2x+1)^2 = (2y-1)(2y+1).$$

* Translation by W. E. Byrne, Virginia Military Institute.

Since $2y-1$ and $2y+1$ are relatively prime, we obtain the following two cases, where m and n are relatively prime odd integers:

$$\text{(a)} \quad \begin{aligned} 2y - 1 &= 3m^2 \\ 2y + 1 &= n^2, \end{aligned} \qquad \text{(b)} \quad \begin{aligned} 2y - 1 &= m^2 \\ 2y + 1 &= 3n^2. \end{aligned}$$

The first case is impossible because we have $n^2-3m^2=2$, which contradicts the fact that 2 is a quadratic non-residue of 3. Putting $m=2k+1$ in the second case we get

$$2y = 4k^2 + 4k + 2 = 2\{(k+1)^2 + k^2\},$$

which proves the theorem.

Also solved by Murray Barbour, W. H. Benson, D. A. Darling, Fritz Herzog, Alexander Kevitz, Roger Lessard, Leo Moser, C. S. Ogilvy, and the Proposer.

Editorial Note. Several solvers obtained explicit forms for the letters involved. These may be indicated briefly as follows.

Let $r(=2+\sqrt{3})$ and $1/r$ be the roots of $R^2-4R+1=0$, and let integers a_n and b_n be given by $2a_n=r^n+r^{-n}$, $2\sqrt{3}b_n=r^n-r^{-n}$. Then $a_n^2-3b_n^2=1$, and it is well known that a_n, b_n, $(n=0, 1, 2, \cdots)$ give the only integral solutions of $a^2-3b^2=1$. Thus the solutions of (1) are given by

$$x_n = (b_{2n+1} - 1)/2, \qquad y_n = a_{2n+1}/2,$$

where alternate a's and b's are taken in order to obtain integral values of x_n, y_n. Direct calculation will verify:

$$y_n = z_n^2 + (z_n + 1)^2, \qquad z_n = (3b_n + a_n - 1)/2,$$

where z_n is always integral.

With initial values $x_0=0$, $y_0=1$, $z_0=0$, $x_1=7$, $y_1=13$, $z_1=2$, it is easy to show that

$$\begin{aligned} x_{n+1} &= 14x_n - x_{n-1} + 6, \\ y_{n+1} &= 14y_n - y_{n-1}, \\ z_{n+1} &= 4z_n - z_{n-1} + 1. \end{aligned}$$

Many of these results have previously been noted, without proof, by V. Thébault [1949, 174].

Factors of $x^{2^n}+y^{2^n}$

4302 [1948, 369]. *Proposed by Joseph Rosenbaum, The Milford School, Connecticut*

Prove that if x and y have no common factor then every odd factor of $x^{2^n}+y^{2^n}$, where n is a positive integer, is of the form $2^{n+1}m+1$.

Solution by J. B. Kelly, University of Wisconsin. Let p be an odd prime such that

$$x^{2^n} + y^{2^n} \equiv 0 \pmod{p}.$$

Since $(x, y) = 1$, we have $x \not\equiv 0$, $y \not\equiv 0 \pmod{p}$. Then

$$(xy^{-1})^{2^n} \equiv -1 \pmod{p},$$

where y^{-1} is the inverse of y in the multiplication group of residues modulo p. Therefore the congruence

$$u^{2^n} \equiv -1 \pmod{p} \tag{1}$$

has a solution. According to Euler's criterion,

$$(-1)^{(p-1)/d} \equiv 1 \pmod{p}, \tag{2}$$

where $d = (p-1, 2^n)$. It follows from (2) that $(p-1)/d$ is even. Then from the definition of d it follows that $p-1$ is divisible by 2^{n+1}, whence

$$p = 2^{n+1}m + 1. \tag{3}$$

Finally, since every product of numbers of the form (3) is also of this form, the proposed theorem is proved.

Also solved by Leo Moser, Robert Seamons, and the Proposer. The Proposer notes the corollary that there are infinitely many primes of the form $2^k m+1$. because every prime factor of

$$(n!)^{2^{k-1}} + 1$$

is of the form $2^k m+1$ and is greater than n.

Coefficients of Bernoulli Polynomials of the Second Kind

4303 [1948, 369]. *Proposed by G. T. Williams, Cambridge, Massachusetts*

If

$$\theta_n = \int_0^1 x(x-1)\cdots(x-n+1)dx,$$

and

$$\phi_n = \int_0^1 x(x+1)\cdots(x+n-1)dx,$$

show that $(-1)^n\theta_n = \phi_n - n\phi_{n-1}$ and find ϕ_n in terms of θ_n.

Solution by Ragnar Dybvik, Levanger, Norway. We have

$$\phi_n - n\phi_{n-1} = \int_0^1 x(x+1)\cdots(x+n-1)dx - n\int_0^1 x(x+1)\cdots(x+n-2)dx$$

$$= \int_0^1 (x-1)x(x+1)\cdots(x+n-2)dx.$$

Setting $x=1-t$, we get

$$\phi_n - n\phi_{n-1} = -\int_1^0 (-t)(-t+1)\cdots(-t+n-1)dt$$

(1)

$$= (-1)^n \int_0^1 t(t-1)\cdots(t-n+1)dt = (-1)^n\theta_n$$

as required. Further

$$\theta_1 = \phi_1 = \int_0^1 x dx = \tfrac{1}{2}$$

and from (1) we have

$$\frac{\phi_i}{i!} - \frac{\phi_{i-1}}{(i-1)!} = \frac{(-1)^i\theta_i}{i!}.$$

Summing for $i=2, 3, \cdots, n$, we get

$$\frac{\phi_n}{n!} - \frac{\phi_1}{1!} = \sum_{i=2}^{n} (-1)^i \frac{\theta_i}{i!},$$

whence

$$\phi_n = n!/2 + n!\sum_{i=2}^{n} (-1)^i\theta_i/i!. \tag{2}$$

Also solved by Joshua Barlaz, J. H. Curtiss, M. S. Klamkin, Max LeLeiko, Roger Lessard, Jules Lieblein, Norman Miller, W. O. Pennell, Robert Seamons, Chih-Yi Wang, and the Proposer.

Editorial Note. LeLeiko points out that $\theta_n = b_n n!$, where b_n is the general coefficient of the Bernoulli polynomial of the second kind. See Jordan, *Calculus of Finite Differences*, 1947, p. 249. Using a result from p. 266 we have

$$\sum_{i=0}^{n-1} (-1)^i \frac{\theta_i}{(n-i)i!} = 0,$$

where $\theta_0 = \phi_0 = 1$, by definition. From this result and (1) we derive

$$\phi_n = n!\sum_{i=0}^{n-1} \frac{\phi_i}{(n-i)(n-i+1)i!}.$$

Wang obtains the following form for ϕ_n,

$$\phi_n = n! \sum_{i=0}^{n} \binom{n-1}{i-1} \theta_i / i!.$$

Partitions

4304 [1948, 369]. *Proposed by H. D. Grossman, New York City*

From the vertices of a regular n-gon three are chosen to be the vertices of a triangle. Prove that the number of essentially different possible scalene triangles is the integer nearest to $(n-3)^2/12$. Compare 3893 [1940, 664].

Solution by Norman Miller, Queen's University, Kingston, Ontario. All the possible types of triangles are included among the $N=(n-1)(n-2)/2$ triangles with a common vertex A. Each of the S types of scalene triangles is counted 6 times in the number N and each of the I types of isosceles, non-equilateral triangles is counted 3 times in N. An equilateral triangle occurs if and only if $n \equiv 0 \pmod 3$. The values of I are readily found on consideration of the possible lengths of the equal sides. The cases are now as follows:

1) $n \equiv \pm 1 \pmod 6$. $I=(n-1)/2$ and $6S+3I=N$.
2) $n \equiv 3 \pmod 6$. $I=(n-1)/2-1$ and $6S+3I+1=N$.
3) $n \equiv \pm 2 \pmod 6$. $I=(n-2)/2$ and $6S+3I=N$.
4) $n \equiv 0 \pmod 6$. $I=(n-2)/2-1$ and $6S+3I+1=N$.

The equations in these four cases yield respectively the following values of S: $(n-3)^2/12-\frac{1}{3}$, $(n-3)^2/12$, $(n-3)^2/12-1/12$, $(n-3)^2/12+\frac{1}{4}$. This proves the theorem.

Note by the Proposer. There is an intimate connection between the present problem 4304, Anning's problem 3893 [1940, 664], and problem 25 from Polya and Szegö, *Aufgaben und Lehrsätze, I*, pp. 4, 157. By reading lattice-point graphs of partitions both horizontally and vertically, we can establish the equality of the number of partitions of n into three $\left\{\begin{array}{l}\text{(a) positive, unequal}\\ \text{(b) positive}\\ \text{(c) non-negative}\end{array}\right\}$ parts and the number of $\left\{\begin{array}{l}\text{(a$'$) positive solutions}\\ \text{(b$'$) solutions with } x, y \text{ non-negative and } z \text{ positive}\\ \text{(c$'$) non-negative solutions}\end{array}\right\}$ of the Diophantine equation $x+2y+3z=n$. A partition of type (a) can be reduced to one of type (b) by subtracting 2 from the largest part and 1 from the next largest, thereby reducing n by 3. The new partition can be again reduced to one of type (c) by subtracting unity from each part, again reducing n by 3. These operations are reversible.

The present problem is equivalent to one of partitions of type (a); 3893 is equivalent to one of partitions of type (b); Polya and Szegö's problem, to one of partitions of type (c). Hence Polya and Szegö's $(n+3)^2/12$ becomes in turn Anning's $n^2/12$ and the present $(n-3)^2/12$.

Also solved by Murray Barbour, John Kelly, Roger Lessard, Leo Moser, and W. E. Patten.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Fluid Dynamics. By V. L. Streeter, New York, McGraw-Hill 1948. 11+263 pages $5.00.

This book is a significant contribution to graduate or senior courses in fluid mechanics. Not requiring the reader to possess a knowledge of mathematics beyond the elementary differential and integral calculus, sufficient advanced mathematical theory is developed in an adequate manner before its use. Before developing details, the author outlines in general statements the theory to be discussed. Careful summaries are drawn which collect the basic assumptions used and the results obtained.

Three-dimensional flow examples are preceded by the foundations of ideal fluid theory. Complex variables are used in the application of conformal mapping to two-dimensional flow. Development of vortex theory is followed by examples. After deriving the Navier-Stokes equations, they are used in examples which include the laminar boundary layer problem.

There is a representative collection of problems. In general the first exercises are routine manipulation. The later ones are of a more difficult nature which when solved contribute significantly to the understanding of the theoretical discussions.

The book is written carefully with an accurate and concise presentation of its material.

E. A. TRABANT

Integraltafel, Erster Teil: Unbestimmte Integrale. By W. Gröbner and N. Hofreiter. Vienna, Springer-Verlag, 1949. 8+166 pages. $5.40.

The authors have taken great pains to make this collection of nearly 1700 indefinite integrals as accurate and complete as possible. The collection contains all the elementary integrals which are commonly found in mathematical handbooks as well as integrals of greater complexity, including elliptic and hyperelliptic integrals and integrals of the elliptic functions of Weierstrass and Jacobi. In many cases (for example, $\int \sin^n x dx$) the authors not only give the customary reduction formulas but also express the integrals in their ultimate series form. The value of the book is enhanced by short summaries of all the methods which are applicable in special cases of frequent occurrence. The handwritten text reproduced on $8\frac{1}{2}\times 12$ pages is particularly attractive and legible. This book should prove of great value to mathematicians, physicists, and engineers.

H. D. LARSEN

Computation Curves for Compressible Fluid Problems. By C. L. Dailey and F. C. Wood. New York, John Wiley and Sons, Inc., 1949. 10+32 pages+32 charts. Not bound. $2.00.

The title of this collection of 32 charts correctly characterizes the book and indicates its practical everyday usage by the engineer. In the introductory 32 pages the authors set out to "explain every relation presented in this text in such a way that anyone using the curves will be able to understand their meaning and limitations without having to refer to any other source." Since acquaintance with detailed thermodynamic relations for a perfect gas and with the circulation theorem is assumed, this commendable effort succeeds only in part.

The charts fall into three categories: I. So-called "Energy Relations" applicable to approximately one-dimensional steady-flow problems with and without heat addition; II. So-called "Plane-Shock Relations" applicable locally across any shock wave; III. "Conical-Shock Relations" summarizing the results of numerical solutions for the steady flow field around an infinite cone placed axially in a supersonic stream.

The basic independent variable in all the charts is the Mach number, local or free-stream. This fact provides a unified treatment but limits somewhat the applicability of the curves since often enough pressure, velocity direction, velocity ratio to the critical speed of sound, M^*, and velocity ratio to the maximum possible speed, $\bar{w}$, are more convenient as independent variables. It is to be hoped that the loose-leaf form of the book will allow additions of other helpful charts found, say, in the publications of the National Advisory Committee for Aeronautics.

As a matter of fact, the theory of Part III uses $\bar{w}$ as a variable, and the absence of an appropriate chart forces the reader repeatedly to raise quantities near unity to the power $\gamma/\gamma-1=7/2$. Also, Chart 4 in Part III is included so that $\bar{w}$ values on the surface of the cone could be expressed in terms of the Mach number on the cone surface. Actually, the relation represented by this chart is a completely general "point relation" valid anywhere and should be included with similar relations in Part I.

In view of the wide range of topics covered, the brief introductory derivations of the various relations understandably falter in places. The questionable derivation of the energy equation will not satisfy the careful student. The derivation of the Prandtl-Meyer relations is awkward and does not have the advantage of physical plausibility. All of the derivations assume steady flow while such a limitation is not necessary for most of the relations in Part II.

In summary, this publication of charts so useful to the engineer working with compressible fluids should be welcome by the profession.

M. V. MORKOVIN

NEW BOOKS RECEIVED

Figures of Fantasy. By A. N. Aheart, New York, Exposition, 1949. 54 pp. $2.00.

Konforme Abbildung. By A. Betz. Berlin, Springer, 1948. 8+539 pp. DM 36.

Introduction to Mathematics. 2nd Edition. By H. R. Cooley, D. Gans, M. Kline, and H. E. Wahlert. Boston, Houghton Mifflin, 1949. 22+170 pp. $4.25.

The Mathematics of Great Amateurs. By J. L. Coolidge. Oxford University Press, 1949. 8+211 pp. $6.00.

Fundamental Algebra with Practical Applications. By R. L. Erickson. New York, McGraw-Hill, 1949. 12+317 pp. $2.80.

Integralgleichungen, Einfughrung in Lehre und Gebrauch. By G. Hamel. Berlin, Springer-Verlag, 1949. 8+166 pp. DM 15.60.

Analytic Geometry. By L. M. Kells and H. C. Stotz. New York, Prentice-Hall, 1949. 8+280 pp. $2.85.

Einführung in die höhere Mathematik. By H. V. Mangoldt. Completely revised and enlarged by K. Knopp. 9th Edition. Stuttgart, Hirzel, 1948. 3 vols. Vol. 1, 16+585 pp. Vol. 2, 16+634 pp. Vol. 3, 16+618 pp. DM 24.80 each.

Practical Mathematics. By C. I. Palmer and S. F. Bibb. Part I, Arithmetic with Applications. 5th Edition. New York, McGraw-Hill, 1949. 12+179 pp. $1.80.

Darstellende Geometrie. (Teubners Mathematische Leitfäden, Volume 2). 2d Edition. By H. von Sanden. Leipzig, Teubner, 1949. 107 pp. $1.65.

Praktische Mathematik. (Teubners Mathematische Leitfäden, volume 44.). By H. von Sanden. Leipzig, Teubner, 1948. 99 pp. $1.00.

Equazioni Differenziali nel Campo Reale, part 2, 2nd Edition. By G. Sansone. Bologna, Zanichelli, 1949. 16+475 pp. 4000 lire.

Tables of the Bessel Functions of the First Kind of Orders, Fifty-two through Sixty-three. (The Annals of the Computation Laboratory of Harvard University, vol. 11.) Cambridge, Harvard University Press, 1949. 12+544 pp. $8.00.

Philosophy of Mathematics and Natural Science. By H. Weyl. Revised and augmented English edition based on a translation by Olaf Helmer. Princeton University Press, 1949. 10+311 pp. $5.00.

An Introduction to the Theory of Mechanics. By K. E. Bullen. Science Press, Sydney, Australia, 1949. xvi+368 pp.

The Axonometric Method of Descriptive Geometry. By W. H. Roever. Lithoprinted by Edwards Brothers, Inc., Ann Arbor, Michigan, 1949. For sale by Washington University Bookstore. 75 pp. $3.00.

Fundamentals of Symbolic Logic. By A. Ambrose and M. Lazerowitz. New York, Rinehart Publishers, 1948. 10+310 pp. $5.00.

Tables of Elementary Functions, 2d Edition. By F. Emde. Leipzig, Teubner, 1948. 12+181 pp. $3.50.

Tables of Higher Functions, 4th Revised Edition. By Jahnke and Emde. Leipzig, Teubner, 1948. 12+300 pp. $3.60.

Mathematics Dictionary, Revised Edition. By G. and R. C. James. New York, Van Nostrand, 1949. 6+432 pp. $7.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1948–49

Mathematics Club, Stanford University

The Stanford *Mathematics Club* held seven meetings during the academic year 1948–49. The following papers were presented:

The solution of Poncelet's problem, by Prof. D. C. Olds

Some fixed point theorems, by Dr. P. Garabedian

Complex moments, by Joe Ullman

Harmonic webs, by Hans Haegi

The gamma function, by John Luke

Space filling curves, by Kenneth Cooke

Progression-free sequences, by Dr. D. C. Spencer

Dr. G. Szego gave a talk on his trip to Europe.

The annual picnic was held at Alum Rock Park.

Officers for 1948–49 were: President, Sherman Lehman; Secretary, Robert Stalley; Treasurer, John Bennett.

Undergraduate Mathematics Club, University of Rochester

The end of the academic year, 1948–49, marks the first year of the newly organized *Undergraduate Mathematics Club* of the University of Rochester. The theme for the year was: that mathematics is not primarily a tool for other sciences but rather a distinct field of investigation.

The papers heard by the members were:

Diophantine equations, by Adnah Kostenbauder

What is a quaternion?, by Nathan Rickless

Almost every number contains an eight, by Norma Schuller

Mathematical fallacies, by Fred Straub

Boolean algebra, by Morten Kenner

Surface area of a cylinder, by Frank Hahn.

Prizes for the best papers were awarded to Adnah Kostenbauder and Fred Straub.

A great deal of interest was aroused by a game analogous to the popular radio game "Twenty Questions." In the game as played by the Mathematics Club, the topic always affiliates itself with something of a mathematical nature, such as geometric figures, equations, sets, *etc.* There is no limit on the number of questions to be asked but the investigating group must arrive at the solution in less than three minutes. The only restriction placed upon the questions is that they must be so directed as to be answered by a *yes* or *no* or in some cases by an indefinite answer of *sometimes* or *possibly*.

The newly elected officers are: Chairman, Fred Straub; Vice-Chairman, Carl Whiteman; Secretary-Treasurer, Frank Hahn; Assistant Secretary-Treasurer, Mary Chapman; Advisors, Dr. Bernstein and Mr. Gunderson.

Kappa Mu Epsilon, Chicago Teachers College

The papers presented to the *Illinois Gamma* chapter of *Kappa Mu Epsilon* during 1948–49 were:

The mystical significance of numbers, by Dorothy Dahlberg

Ramifications in cryptography, by Ramona Goldblatt

Pythagorean number oddities, by Kathryn Graham.

Fifty-seven members attended the annual banquet at which entertainment was furnished by students.

The officers for 1949–50 are: President, Marceline Kostris; Vice-President, Robert McNamara; Secretary, Lois DeGrange; Treasurer, Eleanor Outly; Sponsor, Dr. J. W. Sachs; Corresponding-Secretary, Dr. J. J. Urbancek.

Mathematics Club, Montana State University

The following papers were presented at meetings of the Montana State University *Mathematics Club* the past year:

Bi-products of mathematics, by Dr. A. S. Merrill

The duo-decimal system, by William Shipman

Trisection of angles, by James Wright

The philosophy of mathematics, by Dr. T. G. Ostrom

How to make an atomic bomb, by Dr. C. R. Jeppesen

How to extract cube roots, by Marybelle Fry

Men of mathematics, by Laurence Zimmerman

History of numbers, by Joseph Kratofil.

Informal discussions and refreshments followed the addresses.

A social meeting was held each quarter—a Christmas party in the Fall, Dr. and Mrs. Merrill entertained at a valentine party during the Winter, and the annual picnic with the Chemistry Club was held in the Spring.

The *Pi Mu Epsilon* prizes were awarded at an early meeting.

The officers for next year are: President, Marybelle Fry; Vice-President, Evan Rempel; Secretary-Treasurer, Joseph Kratofil.

Mathematics Club, Immaculate Heart College

The following papers were presented by student members of the club:

Topology, by Marie Kreuper and Dolores Barenberg

The transcendental and imaginary, by Mary Parker

Mathematics as related to other great fields of knowledge, by Mary Gleason

History of linear and quadratic equations, 1550–60, by Mary Dominguez

Historical problems in algebra, by Maureen Brennan, Patricia Hanson, Joan Rae, Glenna Christie, Colleene Harron and Virginia Pritchard

History of the quartic equation and methods of solution, by Dolores Barenberg.

Following each paper a period was devoted to mathematical recreations after which refreshments were served.

Officers for 1948–49 were: President, Doris Mills; Secretary-Treasurer, Marie Kreuper.

Mathematics Club, Purdue University

Monthly meetings were held throughout the year during which the following topics were discussed:

The development of Hilbert spaces, by Roger Burt

The solution of a partial difference equation, by Eugene Usdin

Probability in continuum, by Bertrom Kastant

Probate analysis, by Robert Carson

Summability of divergent series, by Robert Derflinger

Symbolic logic, by Sakari Jutila

The solution of the partial difference equation for Pascal's triangle, by O. Moan

History of the calculus, by Prof. A. Rosenthal.

Mathematics Club, Bowdoin College

The Bowdoin College *Mathematics Club* held nine regular meetings during the academic year 1948–49. During the year the following papers were prepared and discussed:

Perspective triangles, by Dr. S. Hammond

Finite numbers, by Reed Dawson

New aspects of non-Euclidian geometry, by C. Carruthers

Elementary discussion of harmonic functions, by W. Gager

General inversion, by Demetriardies

Dynamic symmetry, by Thorndike

Mathematics of code analysis, by Dr. R. Brumbaugh

The development and significance of the Laplace transform, by D. Union.

Refreshments were served at the last meeting of each semester.

The officers of the club for the year were: President, Charles Carruthers (1st semester), William Gager (2nd semester); Secretary, Alfred Waxler; Treasurer, Donald Union.

Kappa Mu Epsilon, Albion College

The *Michigan Alpha* chapter of *Kappa Mu Epsilon*, Albion College, held seven meetings during the school year 1948–49. The following papers were presented:

Identification of conic sections by invariants, by Ralph Powers

Algebra and the abacus, by Richard Vetter

Contributions to mathematics by ancient Egyptians, by Richard Carver

Fallacies in mathematics, by Wendell Martin

Linkages, by Harold Wakelin

LaGrange and Laplace, by Jean Hayward

Flatland, by Betty Marshall
Proofs of the Pythagorean theorem, by Betty McIllvenan
Non-Euclidean geometry, by Barbara Barnes
Women in mathematics, by Charlotte Phelps.

Officers elected for 1949–50 are: President, Raymond Gillespie; Vice-President, Albert Foster; Secretary-Treasurer, Patricia Collins; Faculty Sponsor, Prof. H. Larsen.

Mathematics Club, Case Institute of Technology

The *Mathematics Club* of Case Institute of Technology enjoyed a year of varied meetings with rather more than usual emphasis on student participation. The following papers were presented:

Plasticity, by Deonisie Trifan
Matrix algebra, by Ernest Leach
The Whitney ham sandwich problem, by Charles Saltzer
Group theory, by Frank Palermo
Exterior ballistics, by Richard Varga
Projective geometry, by Arthur Shaw.

Officers for the year 1948–49 were: President, Ernest Leach; Adviser, Prof. M. Morris.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

GUIDANCE PAMPHLET FOR MATHEMATICS STUDENTS

A guidance pamphlet entitled *Why Study Mathematics* has been published recently by the Canadian Mathematical Congress. It has been prepared for the use of secondary school teachers and students and should be especially helpful to those who are concerned with vocational guidance. Copies may be obtained from the Canadian Mathematical Congress, Engineering Building, McGill University, Montreal, Canada.

PI MU EPSILON JOURNAL

The honorary mathematical fraternity, Pi Mu Epsilon, is now publishing a journal entitled *Pi Mu Epsilon Journal*. The first number indicates that this journal will be of interest to undergraduate as well as graduate members of the fraternity. Inquiries about the journal should be addressed to the Editor, Ruth W. Stokes, 15 Smith Hall, Syracuse University, Syracuse 10, New York.

STATISTICAL SUMMER SESSION IN BERKELEY, CALIFORNIA

Following the established pattern, there will be held this year a Statistical Summer Session at the University of California, Berkeley. The faculty will include W. G. Cochran of Johns Hopkins University, Benjamin Epstein of Wayne University, E. L. Lehmann of the University of California, Paul Lévy of the École Polytechnique, Paris, France, and G. E. Noether of New York University.

Courses will be offered on both the graduate and the undergraduate levels. The graduate courses, all given during the First Summer Session, June 19 to July 29, are meant primarily for students who either have already obtained their Ph.D. degree or are working toward it. No specific prerequisites to graduate courses will be required. The graduate program includes (1) a course on design of experiments and a seminar on analysis of variance by W. G. Cochran, (2) a course on theory of estimation by E. L. Lehmann, and (3) a course and a seminar on random variables and random functions by Paul Lévy.

Inquiries should be addressed to the Office of the Summer Sessions, 1A Administration Building, University of California, Berkeley 4, California.

PERSONAL ITEMS

Cambridge University has announced that the 1949 award of the Adams Prize is divided equally among four men: J. C. Burkill, Fellow and Lecturer of Peterhouse and Lecturer in Mathematics in the University of Cambridge; S. Chandrasekhar, Distinguished Service Professor of the University of Chicago and astrophysicist at the Yerkes Observatory; W. K. Hayman, Lecturer in Mathematics at University College of the Southwest of England, Exeter, and this year Lecturer in Mathematics at Brown University; and J. M. Whittaker, Professor of Mathematics at Liverpool University.

Baylor University reports: Mr. Dale Maness has been appointed to an assistant professorship; Assistant Professor Eugene McLachlan has been granted a year's leave of absence to continue graduate study at Rice Institute.

Catholic University of America announces: Professor O. J. Ramler has been appointed Head of the Department of Mathematics; Mr. Edgar Barrett and Mrs. Paul Nesbeda have been appointed to instructorships.

At Duke University the following appointments to instructorships have been made: Dr. Mary E. Estill of the University of Texas, Mr. L. M. Fulton Jr. and Dr. Walter Rudin, formerly graduate students at Duke University.

Emory University makes the following announcements: Assistant Professor M. B. Sledd has been promoted to an associate professorship; Mr. Earl Carroll, University of Illinois, and Mr. Donald Holmes, Purdue University, have been appointed to instructorships; Professor Douglas Rumble has retired with the title of Professor Emeritus.

Florida State University reports the following: Mr. J. P. Gill formerly at the University of Texas has been appointed to an assistant professorship; Mr. R. W. Bagley and Mr. K. W. Martin have been appointed to instructorships.

At Johns Hopkins University: Dr. Ion Carstoiu of the University of Paris has been appointed to an assistant professorship; Mr. A. C. Smith, formerly of Stonehurst College, Lancashire, England, has been appointed to an instructorship.

Kansas State College announces: Instructor Marilynn Spanglet has been promoted to an assistant professorship; Professor A. E. White is now teaching on a part-time basis.

Lehigh University announces the following promotions and appointments: Assistant Professor F. S. Beale has been promoted to an associate professorship; Assistants B. C. Kenny and Michael Tikson have been promoted to instructorships; Assistant Professor Jacob Samoloff of the University of New Brunswick has been appointed to an instructorship.

The National Bureau of Standards announces: Professor M. R. Hestenes of the University of California at Los Angeles is a visiting staff member at the Institute for the year 1949–1950; Mr. John Todd has been appointed chief of the Computation Laboratory of the Bureau in Washington; Mr. H. E. Salzer, formerly with the New York Unit of the Computation Laboratory, has joined the Washington staff.

North Carolina State College announces the promotions of Assistant Professors C. L. Carroll, Jr. and P. E. Lewis to associate professorships.

Purdue University reports: Associate Professor I. W. Burr has been promoted to a professorship; Dr. E. A. Trabant has been promoted to an assistant professorship; Professor R. J. Duffin, Carnegie Institute of Technology, served as visiting professor for the first semester of 1949–1950; Dr. Jacob Korevaar, Mathematical Center, Amsterdam, Holland, has been appointed Visiting Lecturer for 1949–1950; Dr. Gordon Overholtzer of the University of Kansas and Dr. N. P. Yeardley of the University of Cincinnati have been appointed to assistant professorships; Dr. C. C. Goldman, University of Cincinnati, and Mr. Gustave Rabson of the University of Michigan have been appointed to instructorships; Assistant Professor W. H. Hardman, Acting Professor C. H. Skinner and Instructor J. H. Shock have retired.

Rutgers University announces the promotion of Assistant Instructor G. Y. Cherlin to an instructorship and the appointment of C. N. Campopiano, K. T. Lee and Herman Munick to teaching assistantships.

Syracuse University makes the following announcements: Instructor Eckford Cohen has been appointed to an assistant professorship; Associate Professor Lipman Bers is on leave of absence and is at the Institute for Advanced Study.

At the University of Arizona: Miss Elizabeth A. Button, assistant at the University of Wisconsin, Mr. C. B. Harper, teaching fellow at the University of Michigan, Mr. J. P. Middlekauff, graduate student of Stanford University, and Mr. E. D. Stalley, graduate assistant at Stanford University, have been appointed to instructorships.

The University of Houston reports the following promotions: Instructors

C. P. Benner and R. F. Ray to assistant professorships; Assistant Instructors Dean O. Gray and June P. Wood to instructorships.

The University of Manitoba announces: Professor J. W. Lawson has been appointed Acting Dean of Arts and Sciences; Professor W. H. McEwen has been appointed Dean of the Faculty of Graduate Studies and Research; Mr. H. R. Coish of the National Research Council, Chalk River Development, has been appointed to an assistant professorship.

At the University of New Hampshire: Mr. S. R. Peterson, Mr. F. J. Robinson and Mr. C. W. Schenck have been appointed to instructorships.

The University of Oklahoma makes the following announcements: Associate Professor Casper Goffman has been promoted to a professorship; Mr. P. W. M. John, Mr. D. A. Gorsline, Mr. Gene Levy of the College of the Ozarks and Mr. C. S. Williams have been appointed to instructorships; Mrs. Lyla Tisdale, formerly at Southern Methodist University, has been appointed Special Instructor.

The University of Pennsylvania reports: Dr. Bernard Epstein has been promoted to an assistant professorship; Dr. M. I. Aissen, Stanford University, Dr. H. E. Campbell, University of Wisconsin, and Dr. Anne M. Whitney have been appointed to instructorships.

The University of Rochester announces: Dr. Norman Gunderson has been promoted to an assistant professorship; Professor Waldimir Seidel has returned after a year's leave of absence; Mr. Warren Stenberg of the University of California at Berkeley has been appointed to an instructorship; Mr. Lester Arnold, Mr. David Barton, Mr. A. E. Danese and Mr. John Mairhuber have been appointed to part-time instructorships; Dr. S. L. Crump who is associated with the Atomic Project at the University Hospital is teaching part-time.

The University of Saskatchewan announces the appointments of Dr. Albert Edrei, formerly of Farouk 1st University, Alexandria, Egypt, and Mr. E. I. Gale of the University of New Brunswick as Special Lecturers.

The University of Toronto reports the following: Assistant Professor W. J. R. Crosby of the University of Saskatchewan and Instructor D. A. S. Fraser of Princeton University have been appointed to assistant professorships; Mr. H. L. Seal, actuary, National Life Assurance Company, has been appointed Special Lecturer; Dr. R. G. Stanton, formerly lecturer at the University of Michigan, has been appointed Lecturer.

The University of Utah announces: Dr. R. N. Thomas, formerly of Harvard University and the Institute for Advanced Study, has been appointed to an associate professorship in mathematical astronomy; Miss Rachel S. Hodges of the University of Wisconsin, Mr. R. E. Janssen of the University of Illinois and Mr. H. C. Mayer of the University of Minnesota have been appointed to instructorships; Associate Professor C. J. Thorne has returned after a year's leave of absence; Professor C. R. Wylie Jr. has been elected Chairman of the Mathematics Division of the American Society for Engineering Education for the current year.

At the University of Western Ontario: Professor H. R. Kingston, head of the Department of Mathematics, has been promoted to the position of Dean of the College of Arts and Science; Associate Professor R. H. Cole has returned after a year's leave of absence; Professor Arthur Woods has retired.

The University of Wisconsin reports: Assistant Professor R. H. Bing, now at the University of Virginia, has been promoted to an associate professorship; Assistant Professor R. C. Buck of Brown University has been appointed to an associate professorship; Mr. R. G. D. Steel of Iowa State Teachers College has been appointed to an assistant professorship; Professor H. W. March has retired with the title of Professor Emeritus.

Instructor Tyler Allhands of the State College of Washington has been appointed to an assistant professorship at Pacific University.

Instructor Helmut Aulbach of Syracuse University has been appointed to an assistant professorship at Triple Cities.

Mr. Earle Canfield of Drake University has been promoted to an assistant professorship.

Mr. P. L. Chessin of Columbia University has been appointed to an instructorship at Cooper Union.

Mr. D. M. Clarke has accepted an instructorship at Gonzaga University.

Mr. F. B. Crippen of Rutgers University has been appointed to an instructorship at Fordham University.

Mr. W. S. Doss has been appointed to an instructorship at Springfield College.

Miss Katherine Garland of the University of Denver has been promoted to an instructorship.

Associate Professor R. E. Gilman of Brown University has been promoted to a professorship.

Mr. W. A. Glenn, formerly graduate student at the University of New Brunswick, has been appointed Lecturer at Mount Allison University, New Brunswick, Canada.

Dr. B. H. Gundlach, formerly at Ruston Academy, Havana, Cuba, has been appointed to an assistant professorship at the University of Arkansas.

Mr. Raymond Gutzman of Fenn College has been appointed to an assistant professorship at the Colorado School of Mines.

Dr. H. L. Harter of Purdue University has been appointed to an assistant professorship at Michigan State College.

Dr. T. E. Hull has received an appointment as instructor at the University of British Columbia.

Dr. C. C. Hurd, who has been a research head at Oak Ridge, Tennessee, has accepted an appointment as director of the Applied Science Department of International Business Machines Corporation.

Professor J. L. Jones of the University of Toledo has retired.

Associate Professor W. Kozekiewicz has been appointed to a professorship at the University of Montreal.

Associate Professor Eugene Leimanis of the University of Riga, Latvia, has accepted an appointment as assistant professor at the University of British Columbia.

Professor W. R. Longley of Yale University has been appointed Visiting Professor of Mathematics at Haverford College for the second semester 1949–1950.

Professor W. V. Parker of the University of Georgia has been appointed to the position of Head Professor of Mathematics at Alabama Polytechnic Institute.

Dr. T. D. Reynolds of Duke University is now Director of Instruction, Buncombe County, North Carolina, Schools.

Professor C. K. Robbins of Purdue University has accepted the editorship of the Department of Mathematical Miscellany of the *Mathematics Magazine.*

Professor W. W. Rogosinski of King's College, University of Durham, England, is Visiting Professor of Mathematics at the Oklahoma Agricultural and Mechanical College for the second semester, 1949–1950.

Assistant Professor Andrew Sobczyk of Boston University has been promoted to an associate professorship.

Assistant Professor R. D. Specht of the University of Wisconsin is now with the Rand Corporation.

Assistant Professor Irving Sussman of the University of California at Berkeley has accepted a professorship at St. Mary's College of California.

Associate Professor Ethel Sutherland has been promoted to a professorship at Longwood College, formerly named State Teachers College, Farmville, Virginia.

Dr. Sylvan Wallach of Johns Hopkins University is now with the Westinghouse Company in Pittsburgh.

Professor Alexander Weinstein has accepted a joint appointment by the Department of Mathematics of the University of Maryland and the Institute for Fluid Dynamics and Applied Mathematics.

Instructor G. C. Zader of the Citadel has been promoted to an assistant professorship.

Professor F. F. Decker of Syracuse University died November 28, 1949. He was a charter member of the Association.

Professor A. E. Mallory of Colorado State College of Education died June 1949.

Professor Emeritus Virgil Snyder of Cornell University died January 4, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-THIRD ANNUAL MEETING OF THE ASSOCIATION

The thirty-third annual meeting of the Mathematical Association of America was held at Columbia University, New York City, on Friday, December 30, 1949, in conjunction with the annual meetings of the American Mathematical Society and the American Association for the Advancement of Science. A total of seven hundred and eighty persons were registered at Earl Hall, including the following four hundred and twelve members of the Association:

C. R. ADAMS, Brown University
R. P. AGNEW, Cornell University
L. V. AHLFORS, Harvard University
M. I. AISSEN, University of Pennsylvania
A. A. ALBERT, University of Chicago
R. G. ALBERT, Brown University
C. B. ALLENDOERFER, Haverford College
R. D. ANDERSON, University of Pennsylvania
R. LUCILE ANDERSON, Hunter College
R. G. ARCHIBALD, Queens College
K. J. ARNOLD, University of Wisconsin
L. A. AROIAN, Hunter College
MAX ASTRACHAN, U.S.A.F. Institute of Technology
SILVIO AURORA, Columbia University
FRANK AYRES, Dickinson College
W. L. AYRES, Purdue University
FRANCES E. BAKER, Vassar College
JOSHUA BARLAZ, Rutgers University
P. T. BATEMAN, Institute for Advanced Study
M. R. BATES, Union College
HELEN P. BEARD, Sophie Newcomb College
RALPH BEATLEY, Harvard University
F. S. BECKMAN, Pratt Institute
F. P. BEER, Lehigh University
E. G. BEGLE, Yale University
A. A. BENNETT, Brown University
MARTIN BERMAN, University of Cincinnati
R. R. BERNARD, Yale University
W. W. BESSELL, U. S. Military Academy
BARBARA B. BETTS, D. C. Heath & Company
R. H. BING, University of Virginia
GARRETT BIRKHOFF, Harvard University
ERWIN BISER, Rutgers University
D. W. BLACKETT, Princeton University
ARCHIE BLAKE, Office of Army Surgeon General
W. E. BLEICK, U.S.N. Postgraduate School
H. F. BOHNENBLUST, California Institute of Technology
GRACE L. BOLTON, Barnard College
T. A. BOTTS, University of Virginia
J. G. BOWKER, Middlebury College
C. B. BOYER, Brooklyn College
A. D. BRADLEY, Hunter College
RICHARD BRAUER, University of Michigan
N. A. BRIGHAM, University of Maryland
H. W. BRINKMANN, Swarthmore College
PAUL BROCK, Reeves Instrument Corporation
A. B. BROWN, Queens College
R. H. BRUCK, University of Wisconsin
H. D. BRUNK, Rice Institute
W. J. BRUNS, Syracuse University
C. T. BUMER, Clark University
L. H. BUNYAN, Rutgers University
R. S. BURINGTON, Bureau of Ordnance
L. J. BURTON, Bryn Mawr College
L. E. BUSH, College of St. Thomas
J. H. BUSHEY, Hunter College
JEWELL HUGHES BUSHEY, Hunter College
S. S. CAIRNS, University of Illinois
J. W. CALKIN, Los Alamos Scientific Laboratory
ETHEL B. CALLAHAN, Hartwick College
MILDRED E. CARLEN, Brown University
L. VIRGINIA CARLTON, Northwestern State College
J. O. CHELLEVOLD, Lehigh University
SARVADAMAN CHOWLA, University of Kansas
RANDOLPH CHURCH, U.S.N. Postgraduate School
F. E. CLARK, Tulane University
G. R. CLEMENTS, U. S. Naval Academy
E. J. COGAN, Pennsylvania State College
A. C. COHEN, University of Georgia
H. J. COHEN, Tulane University
L. W. COHEN, Queens College
R. M. COHN, Rutgers University
NANCY COLE, Syracuse University
J. B. COLEMAN, Columbia, S. C.
H. R. COOLEY, New York University
T. F. COPE, Queens College
A. H. COPELAND, University of Michigan
BYRON COSBY, State University of Iowa

R. R. COVEYOU, Oak Ridge National Laboratory
W. H. H. COWLES, Pratt Institute
V. F. COWLING, University of Kentucky
E. H. CRISLER, West Virginia University
E. H. CUTLER, Lehigh University
J. M. DANSKIN, Office of the Chief of Naval Operations
J. E. DARRAUGH, Case Institute of Technology
CONSTANCE H. DAVIS, South Orange, N. J.
D. R. DAVIS, New Jersey State Teachers College
A. H. DIAMOND, Oklahoma A & M College
MARY P. DOLCIANI, Vassar College
H. L. DORWART, Trinity College
R. H. DOWNING, U.S.A.F. Institute of Technology
ARNOLD DRESDEN, Swarthmore College
NELSON DUNFORD, Yale University
P. S. DWYER, University of Michigan
J. N. EASTHAM, Cooper Union
W. F. EBERLEIN, University of Wisconsin
SAMUEL EILENBERG, Columbia University
CAROLYN EISELE, Hunter College
PAUL ERDÖS, University of Illinois
R. W. ERICKSON, Mississippi State College for Women
W. H. FAGERSTROM, City College of the City of New York
F. D. FAULKNER, University of Michigan
J. M. FELD, Queens College
WILLIAM FELLER, Cornell University
W. E. FERGUSON, Connecticut College for Women
F. A. FICKEN, University of Tennessee
N. J. FINE, University of Pennsylvania
C. H. FISCHER, University of Michigan
EDWARD FLEISHER, Brooklyn College
E. E. FLOYD, University of Virginia
TOMLINSON FORT, University of Georgia
R. M. FOSTER, Polytechnic Institute of Brooklyn
J. H. FOUNTAIN, University of Buffalo
J. S. FRAME, Michigan State College
W. M. FRANK, Yeshiva University
ORRIN FRINK, Pennsylvania State College
H. M. GEHMAN, University of Buffalo
ABE GELBART, Syracuse University
B. H. GERE, Hamilton College
F. J. GERST, Loyola University
J. C. GIBSON, Rensselaer Polytechnic Institute
B. P. GILL, City College of the City of New York
R. E. GILMAN, Brown University
J. W. GIVENS, University of Tennessee
A. M. GLEASON, Harvard University
H. E. GOHEEN, Syracuse University
MICHAEL GOLDBERG, Bureau of Ordnance
A. W. GOODMAN, University of Kentucky
D. B. GOODNER, Florida State University
R. D. GORDON, University of Buffalo
W. O. GORDON, Pennsylvania State College
ARTHUR GRAD, Office of Naval Research
H. S. GRANT, Rutgers University
J. B. GREELEY, Utica College
HARRIET M. GRIFFIN, Brooklyn College
GEORGE GROSSMAN, Polytechnic Institute of Brooklyn
C. C. GROVE, City College of the City of New York
H. T. GUARD, Colorado A & M College
R. W. HAMMING, Bell Telephone Laboratories
HELEN J. HAND, D'Youville College
FRANK HARARY, University of Michigan
G. E. HAY, University of Michigan
F. S. HAWTHORNE, Hofstra College
KATHARINE E. HAZARD, New Jersey College for Women
G. A. HEDLUND, Yale University
C. E. HEILMAN, Elizabethtown College
M. H. HEINS, Brown University
G. C. HELME, Pratt Institute
R. G. HELSEL, Ohio State University
A. S. HENDLER, Rensselaer Polytechnic Institute
R. E. HENRY, Newark College, Rutgers University
I. N. HERSTEIN, University of Kansas
M. R. HESTENES, University of California at Los Angeles
T. H. HILDEBRANDT, University of Michigan
J. C. HILFERTY, St. Thomas More High School
L. S. HILL, Hunter College
EINAR HILLE, Yale University
T. R. HOLLCROFT, Wells College
H. K. HOLT, Union College
E. MARIE HOVE, Hofstra College
C. C. HSIUNG, University of Wisconsin
R. C. HUFFER, Beloit College
RALPH HULL, Purdue University
M. GWENETH HUMPHREYS, Randolph-Macon Woman's College
T. R. HUMPHREYS, New Jersey State Teachers College
MILDRED HUNT, Illinois Wesleyan University

SOLOMON HURWITZ, City College of the City of New York
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L. C. HUTCHINSON, Polytechnic Institute of Brooklyn
M. A. HYMAN, Naval Ordnance Laboratory
J. E. IKENBERRY, Madison College
EUGENE ISAACSON, New York University
E. D. JENKINS, Kent State University
WALTER JENNINGS, U.S.N. Postgraduate School
FRITZ JOHN, New York University
R. A. JOHNSON, Brooklyn College
F. E. JOHNSTON, George Washington University
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B. W. JONES, University of Colorado
O. C. JUELICH, Hofstra College
MARK KAC, Cornell University
W. C. KALINOWSKI, St. John's University
AIDA KALISH, Polytechnic Institute of Brooklyn
SIDNEY KAPLAN, Naval Ordnance Laboratory
IRVING KAPLANSKY, University of Chicago
GILBERT KASKEY, University of Delaware
EDWARD KASNER, Columbia University
E. S. KENNEDY, Brown University
D. E. KIBBEY, Syracuse University
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J. R. KLINE, University of Pennsylvania
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J. C. LANZ, Hershey Junior College
E. H. LARGUIER, Spring Hill College
H. D. LARSEN, Albion College
V. V. LATSHAW, Lehigh University
SOLOMON LEFSCHETZ, Princeton University
C. H. LEHMANN, Cooper Union
MARGUERITE LEHR, Bryn Mawr College
H. R. LEIFER, Veterans Administration
MAX LELEIKO, Rutgers University
TADEUSZ LESER, University of Kentucky
CAROLINE A. LESTER, New York State College for Teachers
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A. J. LEWIS, University of Denver
D. C. LEWIS, Johns Hopkins University
SALLY I. LIPSEY, Hunter College
H. D. LIPSICH, University of Cincinnati
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E. R. LORCH, Columbia University
R. B. LOWE, Polytechnic Institute of Brooklyn
C. I. LUBIN, University of Cincinnati
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L. A. MACCOLL, Bell Telephone Laboratories
C. C. MACDUFFEE, University of Wisconsin
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H. M. MACNEILLE, American Mathematical Society
NATHANIEL MACON, University of North Carolina
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FREDERIC H. MILLER, Cooper Union
W. H. MILLS, Yale University
P. D. MINTON, University of North Carolina
H. J. MISER, Headquarters, U.S.A.F.
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HARRIET F. MONTAGUE, University of Buffalo
MABEL D. MONTGOMERY, University of Buffalo
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MARSTON MORSE, Institute for Advanced Study
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D. J. MYATT, Antioch College
D. S. NATHAN, City College of the City of New York
A. B. NEALE, Naval Aircraft Factory, Philadelphia
C. A. NELSON, Rutgers University
F. E. NEMMERS, State University of Iowa

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C. V. NEWSOM, University of the State of New York
HELEN K. NICKERSON, Wheaton College
O. M. NIKODYM, Kenyon College
M. A. NORDGAARD, Upsala College
P. B. NORMAN, Wagner College
G. G. O'BRIEN, Washington Missionary College
EUGENE ODIN, Arma Corporation, New York
L. F. OLLMANN, Hofstra College
ANNE FRANCES O'NEILL, Smith College
E. R. OTT, Rutgers University
J. C. OXTOBY, Bryn Mawr College
F. W. PERKINS, Dartmouth College
C. L. PERRY, University of Arkansas
G. W. PETRIE, Lehigh University
B. J. PETTIS, Institute for Advanced Study
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C. R. PHELPS, Rutgers University
C. G. PHIPPS, University of Florida
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J. C. POLLEY, Wabash College
WALTER PRENOWITZ, Brooklyn College
G. B. PRICE, University of Kansas
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TIBOR RADO, Ohio State University
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GORDON RAISBECK, Bell Telephone Laboratories
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MINA S. REES, Office of Naval Research
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IRVING REINER, University of Illinois
ERIC REISSNER, Massachusetts Institute of Technology
C. N. REYNOLDS, West Virginia
D. P. RICHARDSON, University of Arkansas
MOSES RICHARDSON, Brooklyn College
C. C. RICHTMEYER, Central Michigan College
P. R. RIDER, Washington University
R. F. RINEHART, Research and Development Board
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E. K. RITTER, University of Michigan
J. H. ROBERTS, Duke University
G. N. ROBINSON, New Bedford, Massachusetts
SELBY ROBINSON, City College of the City of New York
V. N. ROBINSON, U. S. Naval Academy
G. B. ROBISON, Cornell University
C. A. ROGERS, Institute for Advanced Study
IRA ROSENBAUM, University of Miami
P. C. ROSENBLOOM, Syracuse University
ARTHUR ROSENTHAL, Purdue University
J. B. ROSSER, National Bureau of Standards
M. F. ROSSKOPF, Syracuse University
CHARLES SALKIND, Polytechnic Institute of Brooklyn
CHARLES SALTZER, Case Institute of Technology
R. G. SANGER, Kansas State College
ARTHUR SARD, Queens College
A. C. SCHAEFFER, Purdue University
R. D. SCHAFER, University of Pennsylvania
ROBERT SCHATTEN, University of Kansas
HENRY SCHEFFE, Columbia University
EDITH R. SCHNECKENBURGER, University of Buffalo
K. C. SCHRAUT, University of Dayton
AUGUSTA L. SCHURRER, University of Wisconsin
ABRAHAM SCHWARTZ, City College of the City of New York
WLADIMIR SEIDEL, University of Rochester
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L. W. SHERIDAN, College of St. Thomas
L. L. SILVERMAN, Dartmouth College
JAMES SINGER, Brooklyn College
SISTER MARY CORDIA, College of Notre Dame of Maryland
SISTER MARY ESTHER, Mundelein College
L. L. SMAIL, Lehigh University
ROBERT EDWARD SMITH, College of William and Mary
W. M. SMITH, Lafayette College
ERNST SNAPPER, Princeton University
W. S. SNYDER, University of Tennessee
ANDREW SOBCZYK, Boston University
ELIZABETH S. SOKOLNIKOFF, University of Wisconsin
J. L. SOLOMON, University of Toronto
D. E. SOUTH, University of Kentucky
W. K. SPEARS, Purdue University
VIVIAN E. SPENCER, U. S. Bureau of the Census
E. R. STABLER, Hofstra College
E. P. STARKE, Rutgers University
J. R. K. STAUFFER, Rhode Island State College
RUTH W. STOKES, Syracuse University

R. R. STOLL, Lehigh University
E. A. STURLEY, Allegheny College
GABOR SZEGO, Stanford University
J. S. TAYLOR, University of Pittsburgh
MILDRED E. TAYLOR, Mary Baldwin College
A. J. TERZUOLI, Polytechnic Institute of Brooklyn
FEODOR THEILHEIMER, Naval Ordnance Laboratory
J. M. THOMAS, Duke University
C. B. TOMPKINS, George Washington University
D. L. THOMSEN, Haverford College
W. J. THRON, Washington University
MICHAEL TIKSON, Lehigh University
G. L. TILLER, Utica College
JOHN TODD, National Bureau of Standards
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J. W. TUKEY, Princeton University
ANNITA TULLER, Hunter College
J. L. ULLMAN, University of Michigan
F. E. ULRICH, Rice Institute
E. P. VANCE, Oberlin College
HENRY VAN ENGEN, Iowa State Teachers College
HELEN E. VAN SANT, Beaver College
S. I. VROOMAN, Rensselaer Polytechnic Institute
G. L. WALKER, Cornell University
S. E. WALKLEY, University of Illinois
J. L. WALSH, Harvard University
R. M. WALTER, Rutgers University
W. G. WARNOCK, Rensselaer Polytechnic Institute
G. C. WEBBER, University of Delaware
MARGARET C. WEEBER, Teachers College of Connecticut
J. V. WEHAUSEN, Office of Naval Research
LOUIS WEISNER, Hunter College
MARIE J. WEISS, Sophie Newcomb College
J. G. WENDEL, Yale University
F. J. WEYL, Bureau of Ordnance
M. E. WHITE, Hunter College
D. E. WHITFORD, Polytechnic Institute of Brooklyn
P. M. WHITMAN, Johns Hopkins University
D. R. WHITNEY, Ohio State University
G. T. WHYBURN, University of Virginia
W. M. WHYBURN, University of North Carolina
D. V. WIDDER, Harvard University
ALBERT WILANSKY, Lehigh University
G. T. WILLIAMS, Long Island, New York
MARY E. WILLIAMS, Skidmore College
FLORENCE A. WIRSCHING, Purdue University
Y. K. WONG, Institute for Advanced Study
MARGARET Y. WOODBRIDGE, Brooklyn College
FRANCES M. WRIGHT, Triple Cities College
MARIE A. WURSTER, Temple University
BERTRAM YOOD, Cornell University
J. W. T. YOUNGS, Indiana University
OSCAR ZARISKI, Harvard University
J. A. ZILBER, Columbia University
H. J. ZIMMERBERG, Rutgers University
ANTONI ZYGMUND, University of Chicago

Sessions of the Association were held on Friday morning and afternoon in Room 301, Pupin Physics Laboratories of Columbia University. President R. E. Langer presided at the morning session and Professor R. M. Foster at the afternoon session. The Program Committee for the meeting consisted of R. M. Foster, chairman, Jewell H. Bushey, T. F. Cope, and Henry Scheffe.

FIRST SESSION OF THE ASSOCIATION

"Mathematical Biology," by Professor Nicholas Rashevsky, University of Chicago.

"Some remarks on Hartogs' Theorem for Several Complex Variables," by Professor W. T. Martin, Massachusetts Institute of Technology.

"The Pupil's Advocate," by Professor Ralph Beatley, Harvard University.

SECOND SESSION OF THE ASSOCIATION

"Some Applications of Matrices in the Theory of Equations," by Professor C. C. MacDuffee, University of Wisconsin.

"Diffusion Processes and the Theory of Evolution," by Professor William Feller, Cornell University.

"The Education of a Scientific Generalist," by Professor J. W. Tukey, Princeton University.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Thursday afternoon in John Jay Hall of Columbia University. Sixteen members of the Board were present. Among the more important items of business transacted were the following:

The Board voted to approve the appointment by President Langer of the following Nominating Committee for 1950: R. M. Foster, chairman, R. L. Jeffery, J. C. Oxtoby.

A ballot by the Board resulted in the reelection of J. F. Randolph as a member of the Finance Committee for 1950–1953.

It was voted to hold the 1950 Annual Meeting at the University of Florida, Gainesville, Florida, on Saturday, December 30, 1950 and to accept invitations for the 1951 Annual Meeting to be held at Brown University, Providence, Rhode Island and for the 1953 Summer Meeting to be held at Laval University, Quebec, Canada. This latter meeting is to be held in conjunction with the Canadian Mathematical Congress and the one hundredth anniversary celebration of the charter of Laval University.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

President Langer presided at the annual business meeting which was held on Friday at 2:30 in Room 301 Pupin Physics Laboratories.

The Secretary announced the results of the balloting for officers, in which 917 votes were cast. L. M. Graves of the University of Chicago was elected First Vice-President for the two year term 1950–1951. M. R. Hestenes of the University of California at Los Angeles and Marie J. Weiss of Sophie Newcomb College were elected Governors for the three-year term 1950–1952.

Upon motion duly seconded and carried unanimously, it was voted that the By-Laws of the Association be amended by adding the following sentence to Article VII, Dues, Section I: "The Board of Governors may authorize the admission to membership of individuals and classes of applicants without payment of the initiation fee."

It was announced that the Board of Governors had authorized remission of the initiation fee for (1) members of the National Council of Teachers of Mathematics and (2) regularly enrolled undergraduate students in colleges and universities of the United States and Canada whose applications for membership are properly certified by the chairman of the appropriate department of mathematics.

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on the morning of Tuesday, December 27 and continued through Thursday afternoon. The Josiah

Willard Gibbs Lecture was delivered by Professor Norbert Wiener of the Massachusetts Institute of Technology on "The Mathematics of Sensory Prosthesis." Professor S. Chowla of the University of Kansas spoke on "The Riemann Zeta and Allied Functions," and Professor L. V. Ahlfors of Harvard University spoke on "The Classification of Open Riemann Surfaces."

Section A of the American Association for the Advancement of Science met on Tuesday afternoon, at which time Professor G. T. Whyburn of the University of Virginia delivered his retiring Vice-Presidential address: "The Open Mapping Medium in Topological Analysis."

During the week of December 26–31, the American Association for the Advancement of Science and its affiliated societies met in New York City, as did the American Economic Association and its affiliated societies. Among the other societies holding meetings in New York City during the week were: The American Statistical Association, The Econometric Society, The Institute of Mathematical Statistics, and The National Council of Teachers of Mathematics.

On Tuesday afternoon the American Statistical Association met at the Biltmore Hotel for a discussion of "The Pre-statistics Mathematics Course." Speakers were Professors Mark Kac of Cornell University and W. G. Madow of the University of Illinois.

Kappa Mu Epsilon, national honorary mathematics fraternity, held a luncheon meeting on Friday in Butler Hall.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the Meeting consisted of E. R. Lorch, chairman, Grace Bolton, T. F. Cope, H. M. Gehman, B. P. Gill, T. R. Hollcroft, S. B. Littauer, Walter Prenowitz.

Hotel headquarters for the mathematical organizations was the Hotel Governor Clinton. Registration and recreation headquarters were situated in the lobby of Earl Hall on the Columbia University campus.

A tea for members of the mathematical organizations and their guests was given by Columbia University on Tuesday afternoon from 4:00 to 6:00 in the Men's Faculty Club.

On Wednesday evening there was a Musicale in the Casa Italiana under the auspices of the Columbia University Music Department. Participants were E. S. Anderson, bass-baritone, and Harold Triggs, pianist.

A tour of the Museum of Modern Art was arranged for the ladies on the afternoon of Thursday.

A joint dinner for members of the mathematical organizations and their guests was held at 6:30 P.M. on Thursday in the Dining Room of John Jay Hall, Columbia University. Professor P. A. Smith acted as toastmaster. Vice-President G. B. Pegram of Columbia University welcomed the guests on behalf of the University. Professor J. L. Walsh described the plans for the 1950 International Congress of Mathematicians. Dr. H. M. MacNeille discussed the functions of his new office as Executive Director of the Society. Professor Jerzy

Neyman spoke on behalf of the Institute of Mathematical Statistics. On behalf of the Association, President R. E. Langer described some of the problems of collegiate mathematics, especially in connection with the training of teachers.

Through a motion prepared by Professor M. R. Hestenes, which was enthusiastically adopted by those present, the mathematical organizations expressed their sincere appreciation to Columbia University and its staff and to all others who assisted in the arrangements for the meetings. Everyone agreed with the concluding sentence of the motion: "From every point of view the meetings have been a complete success."

H. M. GEHMAN, *Secretary-Treasurer*

THE MAY MEETING OF THE KENTUCKY SECTION

The Kentucky Section of the Mathematical Association of America held its annual meeting at Centre College, Danville, Kentucky, on Saturday, May 14, 1949. The Chairman, Professor W. J. Robinson, Centre College, presided.

Sixty-two persons attended the meeting, including the following twenty members of the Association: D. F. Atkins, M. C. Brown, H. H. Downing, Clarence Ford, Aughtum S. Howard, Sister Charles Mary, W. L. Moore, D. J. Myatt, R. S. Park, D. W. Pugsley, F. M. Pulliam, S. L. Riggs, G. G. Roberts, W. J. Robinson, Florence V. Rohde, W. C. Royster, Klaus Schocken, D. E. South, Guy Stevenson, Jean M. Wyre.

The nominating committee presented the following slate of nominees: Chairman, Dr. H. H. Downing, University of Kentucky; Secretary-Treasurer, Dr. Aughtum S. Howard, Kentucky Wesleyan College. These were duly elected.

The following papers were presented at the meeting:

1. *From spherical trigonometry to plane trigonometry*, by Professor H. H. Downing, University of Kentucky.

Professor Downing showed how the formulas relating to a plane triangle could be derived from the formulas relating to a spherical triangle. The vertices of the spherical triangle were held fixed, and the center of the sphere was allowed to recede to infinity. The spherical triangle thus became in the limit a plane triangle, the sides (arcs) and the angles of the spherical triangle became the sides (straight line segments) and angles of the plane triangle. Use was made of simple theorems on limits including

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad \text{and} \quad \lim_{R \to \infty,\, \alpha \to 0} (R\alpha) = C$$

where R = the radius of the sphere, α = a central angle, and C is the chord subtending the angle α.

2. *The fourth dimension and the Bible*, by Professor Beulah Graham, Campbellsville College, introduced by the Secretary.

This paper constituted a review of the book *The Fourth Dimension and the Bible*, by Anthony Granville. It dealt with the concepts of higher spaces and their remarkable agreement with Bible passages, and emphasized the possibility that the higher-dimensional spaces are the heavens to which we ascend after death. Instead of being restricted to three degrees of freedom, we would be able to move in many independent directions.

3. *On pseudo-doubly periodic functions*, by Mr. D. F. Atkins, University of Kentucky.

The paper considers functions of a complex variable, $f(z)$, which satisfy the equations

$$f(z + w_i) = f(z) + h_i(z), \qquad (i = 1, 2). \tag{1}$$

Among others, the following theorems result: If $f_1(z)$ and $f_2(z)$ satisfy (1), then $f_1(z) - f_2(z)$ is an elliptic function. A necessary and sufficient condition for the existence of a solution is that

$$h_1(z + w_2) - h_1(z) = h_2(z + w_1) - h_2(z).$$

The equations (1) are transformed into $\phi(z+1) = \phi(z)$, $\phi(z+w) = \phi(z) + h(z)$, and a $\phi(z)$ which satisfies these equations is written. It is also shown that if $h(z)$ is entire and $\int_0^1 h(z)dz = 0$, the $\phi(z)$ is entire.

4. *The use of the analytical triangle in curve tracing*, by Miss Virginia Baskett, University of Kentucky, introduced by the Secretary.

The speaker discussed the use of the analytical triangle in sketching algebraic curves. Methods for finding the intercepts and asymptotes of the curve were illustrated, and the determination of approximations to the shape of the curve at the origin and at the points at infinity on the axes were discussed.

5. *Sequential analysis*, by Professor Guy Stevenson, University of Louisville.

This paper gave a brief account of the development of the sequential analysis method developed by Wald, Wolfowitz, and others, together with a short account of its principles applied to sampling rechniques.

6. *What are colleges doing to meet the needs of freshmen in mathematics?* by Sister M. Elizabeth Frisch, Villa Madonna College, introduced by the Secretary.

What colleges can do to meet the needs of freshmen in mathematics is a vital question for every mathematician and teacher. Students entering college today do not have the background in mathematics that was required of them twenty years ago. Hence colleges have been required to change the standards and to supply two alternatives: first, use a double-track plan, College Algebra for those whose ability and future outlook already indicate that they should take it, and a five hour course in general mathematics for the rest; and a second alternative, eliminate the requirements of mathematics for all freshmen. Entrance examinations and placement tests are of help in the proper grouping of students.

7. *Some uses of vectors in elementary courses in mathematics*, by Professor W. J. Robinson, Centre College.

This paper discussed the development of elementary courses in mathematics, and in particular, in analytic geometry, by the use of vectors.

SALLIE E. PENCE, *Secretary*

THE MAY MEETING OF THE NEBRASKA SECTION

The twenty-fifth annual meeting of the Nebraska section of the Mathematical Association of America was held at the University of Nebraska in Lincoln, Nebraska, on Saturday, May 7, 1949. Two sessions were held, at which Professor H. L. Rice presided.

Thirty-seven persons attended the meetings, including the following twenty members of the Association: M. A. Basoco, A. K. Bettinger, W. C. Brenke, C. C Camp, F. Marion Clarke, Helen E. Clarkson, H. M. Cox, J. M. Earl, M. G. Gaba, C. B. Gass, R. E. Heath, B. W. Jones, L. M. Larsen, W. G. Leavitt, E.

J. Lowry, Walter Lyche, Florence E. Poole, H. L. Rice, Lulu L. Runge, and C. J. Tegels.

At the business meeting held after the morning session the following were elected as officers for the coming year: Chairman, C. B. Gass, Nebraska Wesleyan University; Vice-Chairman, H. L. Rice, University of Omaha; Secretary, Lulu L. Runge, University of Nebraska. The meetings for next year are to be held on May 6, 1950, at Nebraska Wesleyan University, Lincoln, Nebraska.

Professor B. W. Jones of the University of Colorado was the guest speaker and presented papers at both the morning and afternoon sessions.

The following short papers were presented:

1. *On systems of linear differential equations*, by W. G. Leavitt, University of Nebraska.

Any system of linear differential equations may be written in matric form. The author showed the utility of this form both for its conciseness and for the opportunity it affords of employing many theorems from the abstract theory of matrices. As an illustration, the matrix notation of a linear transformation was given. Certain results were stated, also, from the asymptotic theory of differential equations involving a parameter.

2. *Approximation of a discontinuous function*, by Professor J. M. Earl, University of Omaha.

A general method of least mth power approximation by polynomials to a discontinuous function was illustrated by showing that if the function $f(x)$ has a finite discontinuity at $x=0$, then under proper restrictions on the continuity of $f(x)$ elsewhere in the interval $(-1, 1)$, the magnitude of the expression $x^2|f(x)-P_n(x)|$ does not exceed n^{-1} uniformly on $(-1, 1)$. The polynomial $P_n(x)$ is determined so as to minimize the integral of the square of that expression from -1 to 1. The method applies to the least mth power problem when the function $f(x)$ to be approximated has a finite number of discontinuities and the multiplier x^2 is replaced by a suitable restricted non-negative function.

3. *An application of continued fractions to solutions of equations*, by Prof. B. W. Jones, University of Colorado.

The applicable properties of simple and nearest integer continued fractions were developed. The technique of using such fractions for finding the solutions of equations was explained, and it was shown that use of the nearest integer continued fraction has some advantages over Horner's method.

4. *Simplified procedure for astrofix by commutation*, by Professor O. C. Collins, University of Nebraska, introduced by Prof. H. L. Rice.

Further simplification was reported in the process of obtaining a determinate "fix" from the observed altitudes of two stars. The appropriate intersection of the two circles of equal altitudes was found by computation. The arrangement of the solution is of such simplicity that it will offer an effective alternative to the St. Hilaire intercept procedure as soon as adequate tables of precomputed data are provided. The use of this method might be preferred in any situation where the navigator wished to avoid the plotting of intercepts and of lines of position together with the use of dead reckoning or of an assumed position.

5. *On a special class of topological operators of the form* $t(u \cdot v)$, by Edwin Halfar, University of Nebraska.

It was shown that certain operators of the form $t(u \cdot v)$, where t is of the form dcd, can be expressed as a sum $dcdu+dcdv$.

6. *On automorphisms of extension fields*, by F. Marion Clarke, University of Nebraska.

The partitioning of a finite, separable, normal extension of a ground field which is induced by the group of relative automorphisms of the extension field over the ground field was demonstrated, first with respect to subfields lying between the ground field and the extension field and corresponding to the subgroups of the automorphism group; and second, with respect to mutually exclusive classes of conjugate elements corresponding to co-sets of these subgroups. The classical theory for polynomials in one variable defining the field extension was generalized to polynomials in n variables.

7. *On a theorem of Liouville*, By W. T. Lenser, University of Nebraska, introduced by Professor H. L. Rice.

A theorem of Liouville concerning the integration in finite terms of an elementary function, with certain extensions and applications, was presented.

8. *Some applications of the finite Fourier transformation*, by M. D. Lamoree, University of Nebraska, introduced by Prof. H. L. Rice.

The speaker illustrated the use of the finite Fourier sine transformation by solving a boundary value problem involving one-dimensional flow of heat in a bar.

9. *Mathematical aesthetics*, by F. Marion Clarke, University of Nebraska.

The number ϕ is defined as the positive root of the equation $x^2=x+1$. Its role as an aesthetic measure alike for the art creator and the art critic is traced from the works of the Pythagoreans to those of a modern French school.

10. *Less about more or more about less*, by Professor B. W. Jones, University of Colorado.

The main ideas of this paper was that the primary aim of a teacher of mathematics should be to cultivate understanding and comprehension in his students. The urge should not be to cover a certain amount of material but to instill fundamental ideas. This tendency to teach less and less about more and more should be reversed. In mathematics and in life the emphasis seems to be on techniques—learning how to do things—rather than on appreciating the reasons for things and their meanings. The primary use of applications should be to cultivate this understanding. Let the student know fundamental principles, and he can make his own applications. In applying these ideas the teacher will notice that the student's methods of expression must not be scorned, and that individual differences must be provided for; such a program requires a superior teacher who will develop means of testing for understanding. An experiment for testing the validity of these proposals was suggested.

11. *Analysis of variance*, by Professor C. M. Harsh, Department of Psychology, University of Nebraska, introduced by Professor H. L. Rice.

Preliminary investigations in the sciences often seek evidence as to which of several possible factors is influencing measurements. If data is grouped according to suspected factors, it is easy to show by algebra that the total variance of scores can be analyzed into variance within group and variance between groups. From each component the variance of a parent population is estimated, and the diversity of these estimates is tested by Fisher's F-ratio for samples from a normal population. More complex designs permit similar tests of interaction between several factors.

12. *A norm for frequencies under the new grading system of the University of Nebraska*, by Professor C. C. Camp.

A previous study of all-university grades for 1938–41 predicted the percentages, 2.6, 9.5, 17.2., 20, 18.4, 13.6, 8.7, 5, 4.9 for the new grades 9, 8, 7, 6, 5, 4, 3, 2, 1 which now replace the old grade intervals of 95 or above, 90–94, 85–89, 80–84, 75–79, 70–74, 65–69, 60–64, and *F* (below passing). The group of 40342 grades for the first semester 1948–49 for courses giving from 1 to 5 hours credit showed the actual percentage frequencies of 2.7, 10.5, 17.8, 22.9, 18.4, 12.8, 6.5, 4,4.4, respectively. This distribution was fitted by a Charlier type *A* and Pearson types I, III. The last gave calculated frequencies of 3.3, 9.5, 17.9 21.8, 19.1, 13.7, 7.9, 4, 2.8. If grades are weighted by hours credit the skewness changes from $-.5$ to $-.7$, and the actual frequency percentages become 2.7, 10.4, 17.1, 22.2, 17.8, 13.1, 6.9, 4.6, 5.1, while the average grade changes from 5.45 to 5.38.

LULU L. RUNGE, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at Gustavus Adolphus College in St. Peter, Minnesota, on Saturday, May 8, 1949. Sessions were held in the forenoon, at luncheon, and in the afternoon. Professors H. M. Anderson, J. M. H. Olmsted and W. R. McEwen (Chairman of the Section) presided at the respective sessions.

Seventy-six persons attended the meeting, including the following fifty-two members of the Association: H. M. Anderson, F. J. Arena, H. H. Barnett, J. E. Bearman, S. Louise Beasley, Sister Ada Marie Boehm, K. H. Bracewell, W. H. Bussey, L. E. Bush, R. H. Cameron, E. J. Camp, C. S. Carlson, Elizabeth Carlson, Helen Engebretson, I. C. Fischer, Gladys Gibbens, Sister Mary Seraphim Gibbons, C. H. Gingrich, K. L. Hankerson, W. L. Hart, Charles Hatfield, Jr., W. C. Kalinowski, G. K. Kalisch, P. G. Kirmser, Sister M. Thomas a Kempis Kloyd, W. S. Loud, H. B. MacDougal, Kenneth May, H. C. Mayer, Jr., W. H. McBride, W. R. McEwen, A. G. Montgomery, W. D. Morgan, M. J. Norris, F. R. Ohnsorg, J. M. H. Olmsted, J. C. Peterson, G. C. Priester, Dorothy V. Schrader, Sister M. Leontius Schulte, L. W. Sheridan, F. C. Smith, A. H. Speltz, Esther R. Steinberg, A. G. Swanson, F. J. Taylor, Takashi Terami, Ella Thorp, H. L. Turrittin, O. E. Walder, K. W. Wegner, and Irene L. Wente.

The nominating committee presented the following slate of nominees for the coming year: Chairman, Kenneth May, Carleton College; Secretary, L. E. Bush, College of St. Thomas; Executive Committee, W. R. McEwen, Duluth Branch, University of Minnesota; R. C. Staley, University of North Dakota; Sister Mary Seraphim Gibbons, College of St. Catherine. These were duly elected.

At the business meeting the following resolutions were adopted: (1) That the Chairman of the Section appoint a committee to investigate the possibility of instituting and financing a high school mathematics contest under the sponsorship of the Section; (2) That the Chairman appoint a committee to cooperate with the Minnesota Council of Teachers of Mathematics, and if possible arrange for joint meetings of the Section with the Minnesota Council; (3) That the bylaws of the Section be amended so as to include North Dakota, Manitoba and Saskatchewan in the territory of the Section.

By invitation of the Executive Committee, Professor S. E. Warschawski delivered an address at the morning session. The title of his address was *Geo-*

metrical Aspects of Function Theory. Professor Warschawski discussed some of the classical problems of geometrical function theory, the Riemann mapping theorem, and the uniformization of algebraic functions. L. Bieberbach's variational method for determining the mapping function of a Jordan domain upon a circle by minimizing the area was outlined, and the convergence of the process was indicated. Simple examples illustrating the problem of uniformization of algebraic functions were given, and the use of conformal mapping in the solution of this problem was explained.

The following eight short papers were presented:

1. *An engineering application of a Volterra integral equation*, by Professor P. G. Kirmser, University of Minnesota.

In certain experiments it is sometimes necessary to measure phenomena occurring at some distance from the observer. One method of doing this is to convert the measurement into an electric signal which is then conducted through a wire to an oscilloscope, where the input electric signal is reproduced with some distortion, the magnitude of distortion depending on the conditions under which the oscilloscope is used. Since the observer sees only the distorted response function, he is interested in determining what was actually measured.

The input and response functions are related by a complicated linear differential equation with constant coefficients, the exact form of which it is impractical to obtain. However, if the response to a known input signal is known, the response to an arbitrary input signal can be calculated by superposition. The result is Duhamel's integral. In this case, however, it is the response that is known, and Duhamel's integral becomes a Volterra integral equation, which can be solved by a method of successive approximations.

2. *The probability that* $2+2=4$, by Professor W. S. Loud, University of Minnesota.

This paper deals with rounding-off error in addition. The result of the addition of n numbers is to be rounded off to a given number of decimal places. We ask the probability that the correct result is obtained if the addends are rounded off to that accuracy before addition. The addends are considered to have a uniform distribution. The result is independent of the particular accuracy used, so for purposes of analysis we may use nearest integer accuracy. As an example, for $n=2$, we would ask what is the probability that the sum of two numbers each between 1.5 and 2.5 should be between 3.5 and 4.5; that is, what is the probability that $2+2=4$. This is easily computed by geometric methods, and is found to be 0.75. The result for n numbers is obtained by Fourier transforms, and is $2\pi^{-1}\int_0^\infty (t^{-1}\sin t)^{n+1}dt$.

3. *Heart-shaped curves*, by Professor W. H. Bussey, University of Minnesota.

In the March 21, 1949 issue of *Life* magazine there was an article about Dr. Jekuthiel Ginsberg and *Scripta Mathematica*. One of the illustrations of "beauty in mathematics" was a heart-shaped curve. It has been copied from *Scripta Mathematica* which has taken it from El-Milick's *Elements d'Algebre Ornamentale*. *Life* said that the equation of the curve was $y=\sqrt{x^2}+\sqrt{a^2-x^2}$. This was obviously a misprint. It was corrected in a later issue of the magazine. Mr. Bussey said that, before he read that correction or looked for the correct equation elsewhere, he made two guesses as to what the equation was intended to be. The first one, which turned out to be correct, was $y=\sqrt[3]{x^2}\pm\sqrt{a^2-x^2}$. The graph of this equation looks just about like the ace of hearts of an ordinary deck of playing cards. The second guess, based partly on the assumption that the curve as it appeared in *Life* might not have been carefully drawn, was $y=|x|\pm\sqrt{a^2-x^2}$, with the absolute value of x as an interpretation of the positive square root of x^2. Mr. Bussey showed that

the graph of this equation is a heart-shaped curve; and he called it a "valentine heart" because it is decidedly more like the hearts on commercial valentine cards than is El-Milick's curve or the ordinary ace of hearts. Then he showed how it was related to certain ellipse, and said that the equation and similar equations related to other ellipses might well be used as formulas for the making of valentine hearts of somewhat different shapes.

4. *Some early mathematical and astronomical tables*, by Mr. W. D. Morgan, St. Paul, Minnesota.

The purpose of this paper was to describe briefly some of the mathematical and astronomical tables of the medieval and renaissance periods. The discussion was accompanied by an exhibit of several of these tables, both in type and manuscript. A very interesting item which was shown was Ptolemy's *Table of Chords*, published in the Leichtenstein edition of the *Almagest*, Venice, 1515. This is the earliest trigonometrical table to appear in print. There was also shown a printed edition of the *Alphonsine Tables*, Venice, 1483. This is the first of ten published editions of this work which appeared between the dates 1483 and 1649. A thirteenth century vellum manuscript of the *Toledo Tables* was also displayed. These tables antedated the *Alphonsine Tables* by about two centuries, but were never printed.

5. *The average sum of the digits of integers*, by Professors L. E. Bush and M. J. Norris, College of St. Thomas.

Let $S(N, r)$ be the sum of the digits, for radix r, of the non-negative integers less than N. Let $A(N, r)$ be $S(N, r)/N$. For fixed r, L. E. Bush has proved that $A(N, r)$ is asymptotic to $(r-1)\log N/2\log r$ (this MONTHLY, vol. 47, 1940, pp. 154–156). In this paper it is proved that the graph of $A(N, r)$ never goes above the chords for the curve $y=(r-1)\log N/2\log r$ which join the points whose abscissas are successive powers of r, and never goes below such chords by more than $r/8$. Modifications of the proof yield sharper limits than $r/8$. For example, for $r=10$ the maximum dip below the chords was reduced to 0.8972. Again for $r=10$, by actually locating the value of N between successive powers of 10 at which the maximum dip is obtained, the size of the dip can be shown to be less than 100/117. No better bound is possible.

6. *Axiomatic definitions of determinants*, by Professor G. K. Kalisch, University of Minnesota.

Various axiomatic definitions of determinants are discussed (cf. E. Artin, *Galois Theory*, 2nd ed., p. 11; C. C. MacDuffee, *Vectors and Matrices*, p. 50). In particular, the Weierstrass definition is proved to be equivalent to two coordinate free definitions in terms of linear transformations of a vector space $V_n(F)$ of n dimensions over a field F into itself. The first of the two definitions mentioned is $D(AB)=D(A)D(B)$, $D(aI)=a^nD(I)$ for a in F (this formulation is valid only for algebraically closed F; for general F the extensibility of D over $V_n(F)$ to $\overline{D}$ over $V_n(\overline{F})$ satisfies the equations above). The second definition is as follows: $D(AB)=D(A)D(B)$; $D(a)=a$ if $n=1$, for a in F; $D(A)=D(B)D(C)$ if A is the direct sum $B+C$. In both formulations, "trivial" solutions ($D=0$ and $D\equiv 1$) are to be excluded. In conclusion it is shown that the existence of a determinant function over $V_n(F)$ if F is a skew field implies the commutativity of F.

7. *Transformations in elementary analytic geometry*, by Professor Kenneth May, Carleton College.

The idea of a transformation of coördinates or a change of variables is fundamental in analytic geometry, and indeed in all analysis, but in the elementary courses it appears rather late as a device for graphing the general conic. Since the formulas for translation follow immediately from the definition of coördinates, and those for rotation require nothing more than trigonometry, transformations could be introduced early in the course and used to simplify proofs and deepen the student's appreciation of the subject. For example, it is easy to prove that by appropriate transla-

tion and rotation any linear equation in x and y can be reduced to either $x=0$ or $y=0$. Then its locus is a straight line, since it is congruent to an axis. This observation indicates that the usual proofs based on the constancy of the slope are redundant. Having proved by transformations that the locus of a linear equation is a straight line, it can easily be shown that any straight line has an equation of this form and that the slope of a straight line is a constant. Of course, the treatment by transformations brings out automatically the relations between the coefficients of the equation and the position of the line.

8. *A graphic approach to a problem in multiple regression*, by Professor L. W. Sheridan, St. Thomas College.

A variable X admitting to one of two alternatives, may be estimated graphically by plotting pairs of observed values of the independent variables, and indicating which of the two alternatives occurred. The data in each such scattergram may then be grouped into cells having any convenient number of observations, and probabilities computed for each cell. Based on the observed probabilities, contour lines then drawn give secondary variables, such as $Y_1=f(X_1, X_2)$, $Y_2=f(X_3, X_4)$, etc. A single variable is obtained by plotting a pair of Y values in the same manner, and continuing the process. This single variable, when plotted against observed values of X, and represented by a smooth curve, may be used to estimate the value of X sought in the problem. An advantage of this method is that it gives a measure of the probability of occurrence of the event.

L. E. BUSH, *Secretary*

CALENDAR OF FUTURE MEETINGS

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.

INDIANA, Wabash College, Crawfordsville, April 29, 1950.

IOWA, State University of Iowa, Iowa City, April 21–22, 1950.

KANSAS, April 22, 1950.

KENTUCKY, University of Kentucky, Lexington, April 29, 1950.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Richmond, May, 1950.

METROPOLITAN NEW YORK, City College, April 1, 1950.

MICHIGAN, University of Michigan, Ann Arbor, March 25, 1950.

MINNESOTA, Macalester College, St. Paul, May 6, 1950.

MISSOURI, Washington University, St. Louis, March 24–25, 1950.

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.

NORTHERN CALIFORNIA

OHIO, Denison University, Granville, April 22, 1950.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June, 1950.

PHILADELPHIA

ROCKY MOUNTAIN, University of Denver, April 28–29, 1950.

SOUTHEASTERN, University of Florida, Gainesville, April 7–8, 1950.

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950.

SOUTHWESTERN, Spring, 1950.

TEXAS, Abilene, April 14–15, 1950.

UPPER NEW YORK STATE, Syracuse University, April 22, 1950.

WISCONSIN, Marquette University, Milwaukee, May 13, 1950.

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